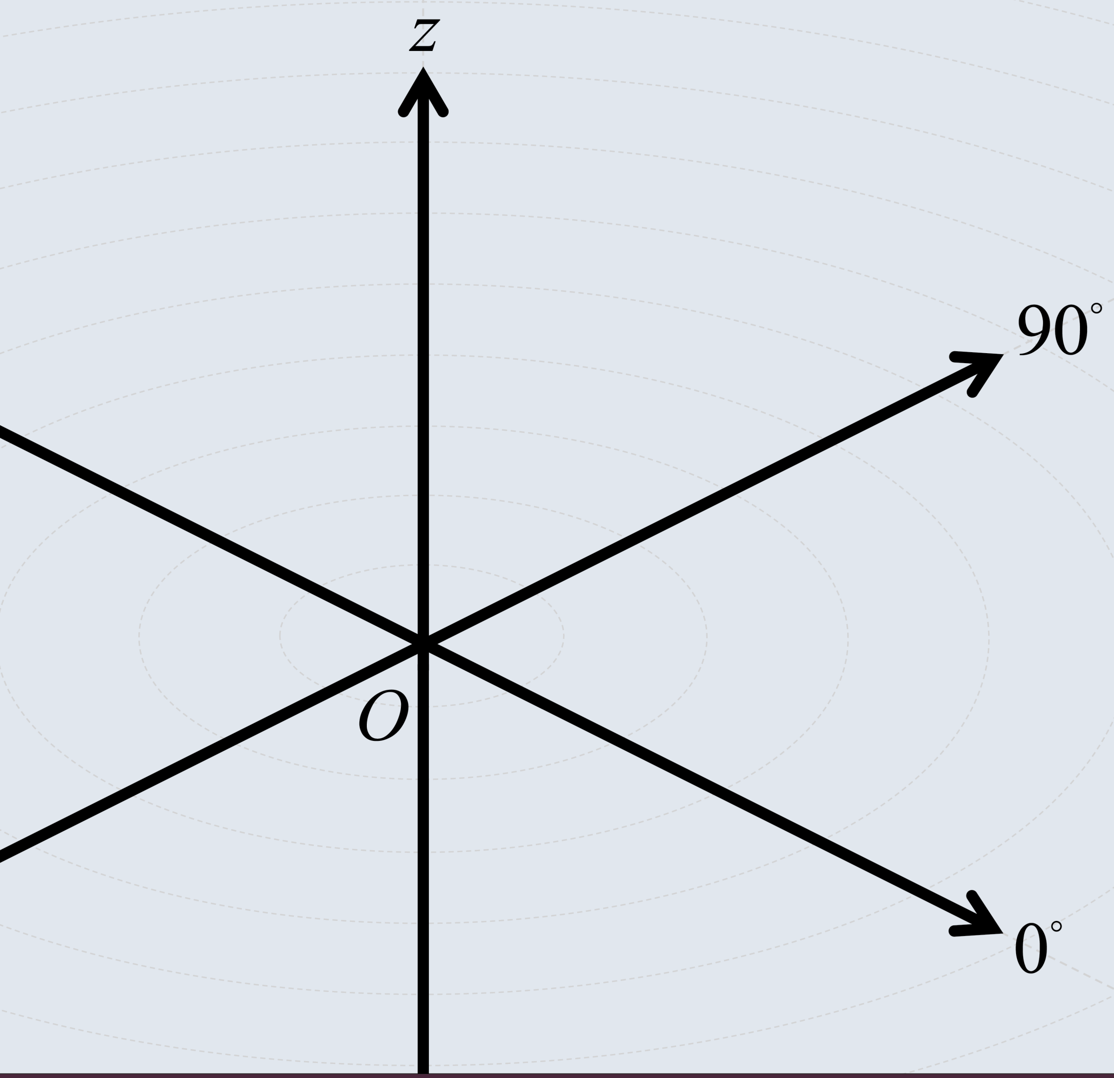
YOUR PRACTICE SET

APPLICATIONS AND INTERPRETATION FOR IBDP MATHEMATICS

FOR HL STUDENTS

Book 2



Stephen Lee

Michael Cheung

Balance Lee

- Compulsory Topics for MAI HL Students
- 80 Example Questions + 320 Intensive Exercise Questions
- Comprehensive Paper 3 Analysis and Practice Questions
- Holistic Exploration on Assessment-styled Questions

SE PRODUCTION LIMITED

Your Practice Set

Applications and Interpretation for IBDP Mathematics Book 2

(For HL Students Only)

Stephen Lee   Michael Cheung   Balance Lee

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Your Practice Set
Applications and Interpretation for IBDP Mathematics
Book 2

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Contents

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Mr. Balance Lee received his Bachelor of Science in Risk Management Science from the Chinese University of Hong Kong, as well as the Master of Statistics in the University of Hong Kong. He has more than 10 years of experience in teaching students from various curricula notably the IBDP and the A level Mathematics syllabuses, including group courses conducted in English. He is currently a tutor mainly for IBDP Mathematics, and at the same time an examiner from the International Baccalaureate Organization, and keeping updated on the syllabus change in Mathematics.

Foreword

People in this world have different views on academic success. Some people think that academic success is measured by scores on examinations, while some may think that it should be measured by the happiness in learning. From our point of view, academic success is that students can learn in an effective way and have enjoyment in the learning process. Students can find learning interesting and have motivation if the learning process is effective, and thus learning becomes enjoyable and the chance of getting good academic results will be greater.

In preparing this book, our team was guided by our experience and interest in teaching IBDP Mathematics. This book is designed to help students to have a good preparation in the brand new challenging two-year International Baccalaureate Diploma Program. This book helps students to review all important concepts in Analysis and Approaches, and helps students to understand how to start to answer a question and get familiar with assessment-styled questions. No doubt, this book can help you achieving high exam scores in IBDP Mathematics. By going through this book, you will find that the questions can help you to answer the structured questions confidently.

To sum up, this book is not only to be a successful practice source, but also to serve as valuable resource for students of each area.

SE Production Team

Updates

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More Recommendations

Your Practice Set – Analysis and Approaches for IBDP Mathematics Book 1

- Common and compulsory topics for both MAA SL and MAA HL students
- 100 example questions + 400 intensive exercise questions in total
- 375 short questions + 125 structured long questions in total
- Special GDC skills included
- Holistic exploration on assessment styled questions
- QR Codes for online solution

Your Practice Set – Analysis and Approaches for IBDP Mathematics Book 2

- Compulsory topics for MAA HL students
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- Common and compulsory topics for both MAI SL and MAI HL students
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Ways to Use This Book

SUMMARY POINTs	Checklist of the concepts of a particular topic for students
Paper 1 Questions	Short questions, usually 4 to 8 marks each
Paper 2 Questions	Structured questions, usually 12 to 20 marks each
[2]	Number of marks for a question
M1	A mark is assigned when the corresponding method is clearly shown
(M1)	A mark is assigned when the corresponding method is not clearly shown but is shown in the following correct working
A1	A mark is assigned when the correct answer is clearly shown
(A1)	A mark is assigned when the correct answer is not clearly shown but is shown in the following correct working
R1	A mark is assigned when the reasoning statement is clearly shown
AG	No mark is assigned as the final step (usually would be answer) is already given from the question

GDC Skills

Some implicit skills of TI-84 Plus CE that you might not heard before

Scenario 1: Solving $f(x)=g(x)$ in Functions

Step 1: Set $f(x)-g(x)=0$

Step 2: Input $\mathrm{Y}_1=f(x)-g(x)$ in the graph function

Step 3: Set the screen size from window

- ✓ $x\min$ and $x\max$: You can refer to the domain given in the question
- ✓ $y\min$ and $y\max$: You can set $y\min=-1$ and $y\max=1$ if you wish to find the x-intercept only

Scenario 2: Finding the number of years, n, when $f(x)=g(x)$ is in the exponent of an exponential model, in Arithmetic Sequences / Geometric Sequences / Logarithmic Functions

Step 1: Set the right-hand-side of the expression to be zero

Step 2: Input $\mathrm{Y}_1=$ the left-hand-side of the expression in the graph function

Step 3: Set the screen size from window

- ✓ $x\min$: You can set $x\min=0$ as n represents the number of years which must be a positive integer

Scenario 3: Finding the x-intercept from the window

- ✓ Assume that the domain is $0\le x\le 100$, and it is clearly shown that the curve cuts the x-axis once only on the left part of the screen
- ✓ You can set the left bound and the right bound to be 0 and 50 respectively to find the x-intercept efficiently, as 50 is the midpoint of the x-axis

Scenario 4: Finding an unknown quantity from the TVM Solver

N = 5 I% = 6 PV = −24000 PMT = 0 FV = ? P / Y = 1 C / Y = 1 PMT : END	→	N = 5 I% = 6 PV = −24000 PMT = 0 FV = 0 P / Y = 1 C / Y = 1 PMT : END

✓ You can set the unknown quantity to be zero in order to execute the program. In the above example, the future value of a compound interest problem is going to be found. You can set FV to be zero and then choose tvm_FV to calculate the future value.

Scenario 5: Finding an area under a curve and above the x-axis

✓ Apart from using the function [MATH] [9], you can sketch the curve and use the function [2nd] [trace] [7], and then set the lower limits and the upper limits.

Scenario 6: Finding probabilities in a Binomial distribution, in the form $\mathrm{P}(X < \text{or} > \text{or} \geq c)$

✓ You need to change the probability to the form $\mathrm{P}(X \leq C)$, and then use the function [2nd] [vars] [B] to choose binomcdf.

Chapter

1

Functions

SUMMARY POINTs

✓ $f \circ g(x) = f(g(x))$: Composite function when $g(x)$ is substituted into $f(x)$

✓ Steps of finding the inverse function $y = f^{-1}(x)$ of $f(x)$:

1. Start from expressing y in terms of x
2. Interchange x and y
3. Make y the subject in terms of x

✓ Properties of $y = f^{-1}(x)$:

1. $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
2. The graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ about $y = x$

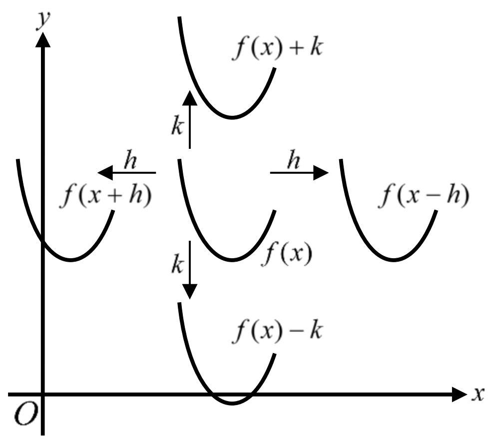
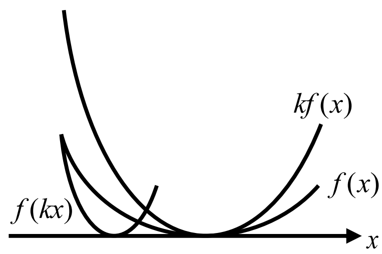
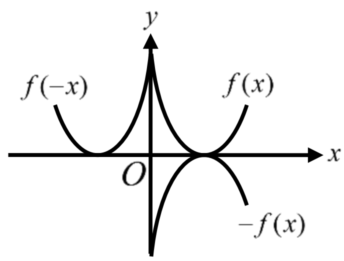
✓ $f^{-1}(x)$ exists only when $f(x)$ is one-to-one in the restricted domain

✓ Variations:

1. $y = kx, k \neq 0$: y is directly proportional to x
2. $y = \frac{k}{x}, k \neq 0$: y is inversely proportional to x

SUMMARY POINTs

✓ Summary of transformations:

	$f(x)\to f(x)+k$: Translate upward by k units $f(x)\to f(x)-k$: Translate downward by k units $f(x)\to f(x+h)$: Translate to the left by h units $f(x)\to f(x-h)$: Translate to the right by h units
	$f(x)\to kf(x)$: Vertical stretch of scale factor k $f(x)\to f(kx)$: Horizontal compression of scale factor k
	$f(x)\to -f(x)$: Reflection about the x-axis $f(x)\to f(-x)$: Reflection about the y-axis

✓ Properties of rational function $y=\dfrac{ax+b}{cx+d}$:

1. $y=\dfrac{1}{x}$: Reciprocal function
2. $y=\dfrac{a}{c}$: Horizontal asymptote
3. $x=-\dfrac{d}{c}$: Vertical asymptote

Solutions of Chapter 1

1 Paper 1 – Composite and Inverse Functions

Example

Let $f(x)=3x+4$ and $g(x)=7x^2-1$.

(a) Find $f^{-1}(x)$.

[3]

(b) (i) Write down $f(2)$.

(ii) Hence, find $(g\circ f)(2)$.

[3]

Solution

(a) $y=3x+4$

$\Rightarrow x=3y+4$ (M1) for swapping variables

$x-4=3y$

$y=\frac{x-4}{3}$ (A1) for changing subject

$\therefore f^{-1}(x)=\frac{x-4}{3}$ A1

[3]

(b) (i) 10 A1

(ii) $(g\circ f)(2)=g(10)$

$(g\circ f)(2)=7(10)^2-1$ (A1) for substitution

$(g\circ f)(2)=699$ A1

[3]

Exercise 1

1. Let $f(x) = 8x - 1$ and $g(x) = x^2 - 5$.

(a) Find $f^{-1}(x)$.

[3]

(b) (i) Write down $g(5)$.

(ii) Hence, find $(f \circ g)(5)$.

[3]

2. Let $f(x) = 2x - 3$ and $g(x) = (x+5)^2$.

(a) Find $f^{-1}(x)$.

[3]

(b) (i) Write down $f(-2)$.

(ii) Hence, find $(g \circ f)(-2)$.

[3]

3. Let $f(x) = \sqrt{x+4}$, for $x \geq -4$.

(a) Find $f^{-1}(4)$.

[3]

(b) Let g be a function such that $g(96) = 7$ and g^{-1} exists for all real numbers.

(i) Write down $g^{-1}(7)$.

(ii) Hence, find $(f \circ g^{-1})(7)$.

[3]

4. Let $f(x) = \sqrt{2x-1}$, for $x \geq \frac{1}{2}$.

(a) Find $f^{-1}(3)$.

[3]

(b) Let g be a function such that g^{-1} exists for all real numbers. Given that $g\left(\frac{3a+1}{2}\right) = 2$, where a is a constant, find $(f \circ g^{-1})(2)$, give the answer in terms of a.

[3]

2 Paper 1 – Restricted Domains for Inverse Functions

Example

The function f is defined as $f(x)=(x-5)^2+3$. The domain of f is restricted as $\{x : x \le k\}$ such that f^{-1} exists.

(a) State the range of f.

[1]

(b) Write down the maximum value of k.

[1]

(c) Find the expression of f^{-1}.

[3]

Solution

(a) $\{y : y \ge 3\}$ A1

[1]

(b) 5 A1

[1]

(c) $y=(x-5)^2+3$

$\Rightarrow x=(y-5)^2+3$ (M1) for swapping variables

$(y-5)^2=x-3$

$y-5=\sqrt{x-3}$

$y=\sqrt{x-3}+5$ (M1) for valid approach

$\therefore f^{-1}(x)=\sqrt{x-3}+5$ A1

[3]

Exercise 2

1. The function f is defined as $f(x)=-(x+7)^2+10$. The domain of f is restricted as $\{x : x \ge k\}$ such that f^{-1} exists.

(a) State the range of f.

[1]

(b) Write down the minimum value of k.

[1]

(c) Find the expression of f^{-1}.

[3]

2. The function f is defined as $f(x)=x^2-8x+40$. The domain of f is restricted as $\{x: x\le c\}$ such that f^{-1} exists.

(a) Express $f(x)$ in the form $f(x)=a(x+h)^2+k$, where a, h, $k\in\mathbb{R}$.

[2]

(b) Hence, write down the maximum value of c.

[1]

(c) Find the expression of f^{-1}.

[3]

3. The function f is defined as $f(x)=4(x+h)^2$, $h<15$. It is given that the graph of f passes through $(-14, 4)$.

(a) Find h.

[2]

The domain of f is restricted as $\{x: x\ge k\}$ such that f^{-1} exists.

(b) Write down the minimum value of k.

[1]

(c) Find the expression of f^{-1}.

[3]

4. The function f is defined as $f(x)=(x-1)^2(x-5)^2$. The domain of f is restricted as $\{x: x\le k\}$ such that f^{-1} exists.

(a) Write down the coordinates of the local minimum of f which is closest to the origin.

[1]

(b) Hence, write down the maximum value of k.

[1]

f can also be expressed as $f(x)=((x-3)^2-4)^2$.

(c) Find the expression of f^{-1}.

[3]

3 Paper 1 – Transformations of Quadratic Functions

Example

A quadratic function f is given by $f(x)=(x+3)^2-1$.

Let $g(x)$ be another quadratic function. The graph of g is obtained by a reflection of the graph of f in the x-axis.

(a) Find $g(x)$, giving the answer in the form $a(x+m)^2+n$.

[2]

Let $h(x)=-(x+5)^2+19$. The graph of h is obtained from g by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(b) Find the values of

(i) p;

(ii) q.

[4]

Solution

(a) $g(x)=-f(x)$ (M1) for valid approach

$g(x)=-((x+3)^2-1)$

$g(x)=-(x+3)^2+1$ A1

[2]

(b) (i) $-3+p=-5$ (M1) for translation

$p=-2$ A1

(ii) $1+q=19$ (M1) for translation

$q=18$ A1

[4]

Exercise 3

1. A quadratic function f is given by $f(x) = -(x-3)^2 - 1$.

Let $g(x)$ be another quadratic function. The graph of g is obtained by a reflection of the graph of f in the x-axis.

(a) Find $g(x)$, giving the answer in the form $a(x+m)^2 + n$.

[2]

Let $h(x) = (x-1)^2 - 5$. The graph of h is obtained from g by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(b) Find the values of

(i) p;

(ii) q.

[4]

2. A quadratic function f is given by $f(x) = (x-8)^2 - 6$.

Let $g(x)$ be another quadratic function. The graph of g is obtained by a reflection of the graph of f in the y-axis.

(a) Find $g(x)$, giving the answer in the form $a(x+m)^2 + n$.

[2]

Let $h(x) = (x+8)^2 + 200$. The graph of h is obtained from g by a translation of $\begin{pmatrix} p \\ q \end{pmatrix}$.

(b) Find the values of

(i) p;

(ii) q.

[4]

3. A quadratic function f is given by $f(x) = -(x-1)^2 + 4$.

Let $g(x)$ be another quadratic function. The graph of g is obtained from f by a translation of $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

(a) Find $g(x)$, giving the answer in the form $a(x+m)^2 + n$.

[3]

Let $h(x) = -3x^2 + 3$. The graph of h is obtained from g by a vertical stretch of scale factor r.

(b) Find the value of r.

[2]

4. A quadratic function f is given by $f(x) = x^2 + 4x + 6$.

Let $g(x)$ be another quadratic function. The graph of g is obtained from f by a vertical stretch of scale factor 5.

(a) Find $g(x)$, giving the answer in the form $ax^2 + bx + c$.

[2]

(b) Write down the coordinates of the vertex of $g(x)$.

[2]

Let $h(x) = 5(3x+2)^2 - 2$. The graph of h is obtained from g by a horizontal stretch of scale factor $\frac{1}{p}$ followed by a translation of $\begin{pmatrix} 0 \\ q \end{pmatrix}$.

(c) Find the values of

(i) p;

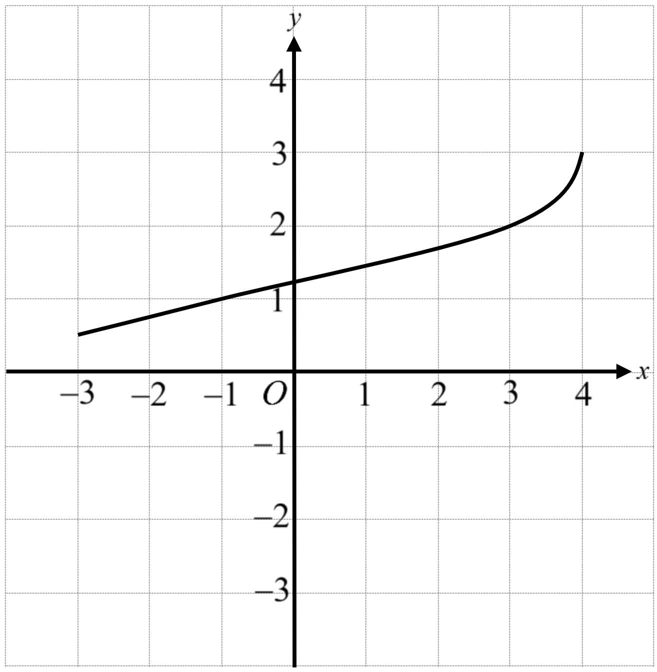
(ii) q.

[4]

4 Paper 1 – Reflections about Coordinate Axes

Example

The following diagram shows the graph of a function f.



(a) Find $f^{-1}(2)$.

[2]

(b) Find $(f \circ f)(4)$.

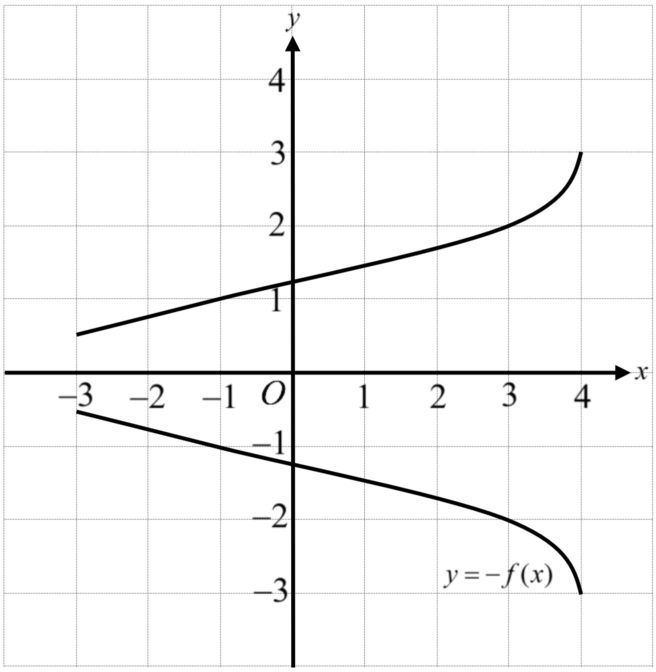
[3]

(c) On the same diagram, sketch the graph of $y = -f(x)$.

[2]

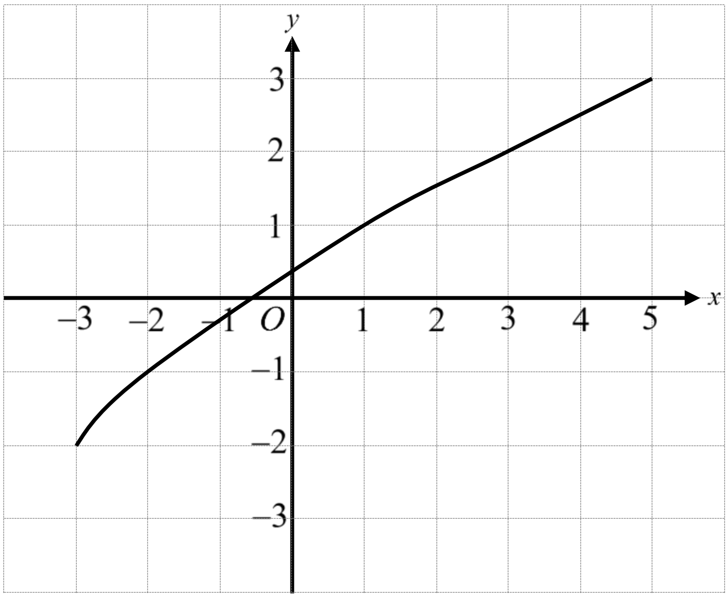
Solution

(a)	$f(3) = 2$	(M1) for correct approach
	$\therefore f^{-1}(2) = 3$	A1
		[2]
(b)	$f(4) = 3$	(M1) for correct approach
	$(f \circ f)(4) = f(3)$	(A1) for composite function
	$(f \circ f)(4) = 2$	A1
		[3]
(c)	For correct y-intercept	A1
	For any two correct points from $(-1, -1)$, $(3, -2)$ and $(4, -3)$	A1
		[2]



Exercise 4

1. The following diagram shows the graph of a function f.



(a) Find $f^{-1}(-2)$.

[2]

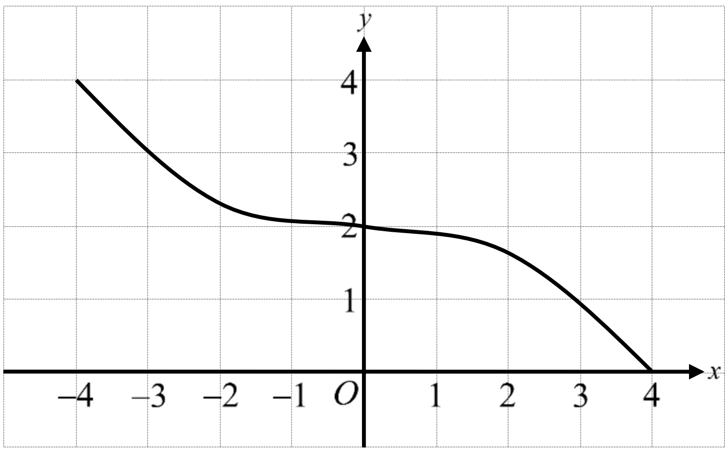
(b) Find $(f \circ f)(5)$.

[3]

(c) On the same diagram, sketch the graph of $y = -f(x)$.

[2]

2. The following diagram shows the graph of a function f.



(a) Find $f^{-1}(2)$.

[2]

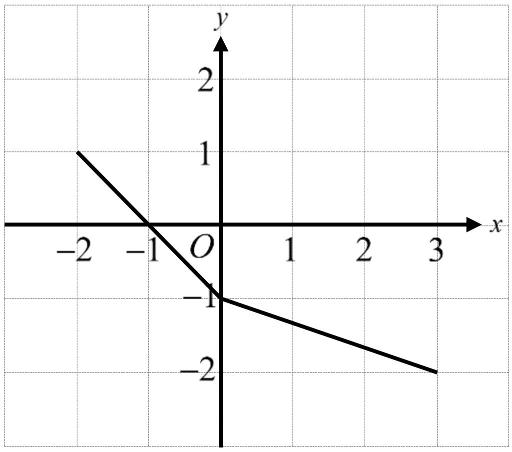
(b) Find $(f \circ f)(4)$.

[3]

(c) On the same diagram, sketch the graph of $y = f(-x)$.

[2]

3. The following diagram shows the graph of a function f.



(a) Find the range of f^{-1}.

[2]

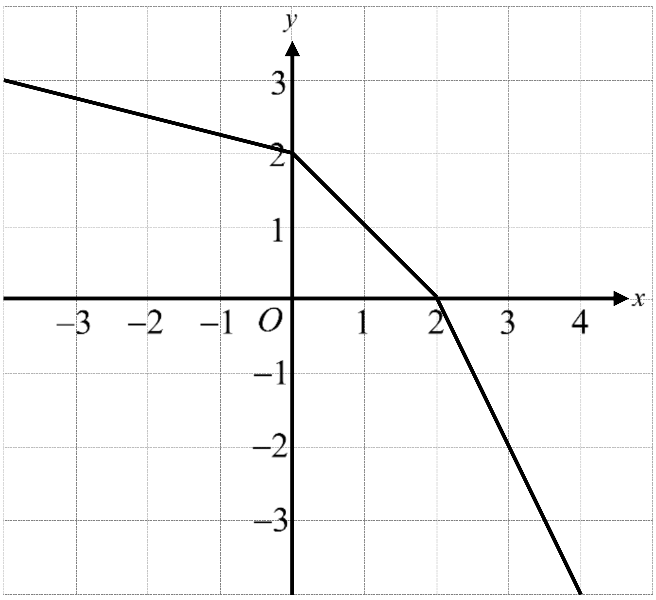
(b) Find $(f^{-1} \circ f^{-1})(1)$.

[3]

(c) On the same diagram, sketch the graph of $y = -f(x)$.

[2]

4. The following diagram shows the graph of a function f.



(a) Find the domain of f^{-1}.

[2]

(b) Find $(f^{-1} \circ f^{-1})(3)$.

[3]

(c) On the same diagram, sketch the graph of $y = f(-x)$.

[2]

5 Paper 1 – Variations

Example

Let $\$C$ be the cost of manufacturing a cubical block of side x cm. It is given that C is directly proportional to x, and the cost of manufacturing a cubical block of side 10 cm is \$35.

(a) Express C in terms of x.

[2]

(b) Write down the cost of manufacturing a cubical block of side 6 cm.

[1]

Suppose that the extra cost \$12 is taken into account.

(c) Find the length of the side of a cubical block with cost \$40.

[2]

The cost factor r is defined as $r=\sqrt{0.875C}$.

(d) Express r in terms of x.

[1]

Solution

(a) $C=kx$, where $k\neq 0$ (M1) for valid approach

$35=10k$

$k=3.5$

$\therefore C=3.5x$ A1

[2]

(b) \$21 A1

[1]

(c) $40=3.5x+12$ (M1) for setting equation

$28=3.5x$

$x=8$

Thus, the required length is 8 cm. A1

[2]

(d) $r=1.75\sqrt{x}$ A1

[1]

Exercise 5

1. In a factory, the production cost of a carpet of perimeter s metres is $\$C$. It is given that C is directly proportional to s, and the production cost of a carpet of perimeter 12 metres is $150.

(a) Express C in terms of s.

[2]

(b) Write down the production cost of a carpet of perimeter 26 metres.

[1]

(c) Find the production cost of a circular carpet with radius 5 metres.

[2]

The cost factor r is defined as $r = \left(\frac{16}{3125}C\right)^{\frac{1}{3}}$.

(d) Express r in terms of s.

[1]

2. Let $\$P$ be the price of a tetrahedron model of surface area of $A\text{ cm}^2$. It is given that P is inversely proportional to A. When $A = 9$, $P = 20$.

(a) Express P in terms of A.

[2]

(b) Write down the price of a tetrahedron model of surface area of 75 cm^2.

[1]

(c) Interpret the condition on the price of a tetrahedron model of a large surface area.

[1]

The straight line graph of $\ln P$ versus $\ln A$ is sketched.

(d) Write down the gradient of the graph.

[1]

3. The weight of a plate of perimeter l centimetres is W grams. It is given that W varies directly as $\sqrt{l}$. When $l = 256$, $W = 288$.

(a) Express W in terms of l.

[2]

(b) Write down the perimeter of a plate of weight 378 grams.

[1]

The graph of W is transformed to the new graph of $W = 36\sqrt{l+3}$ by two transformations.

(c) Describe geometrically for the two transformations.

[2]

4. It is given that $f(x)$ varies inversely as x^2. Suppose that $f(4) = 0.375$.

(a) Express $f(x)$ in terms of x.

[2]

(b) Write down the vertical asymptote of the graph of $f(x)$.

[1]

The graph of $f(x)$ is translated to the right by 3 units, followed by a vertical stretch of scale factor 4 to form a new function $g(x)$.

(c) Find the expression of $g(x)$.

[2]

Chapter

2

Exponential and Logarithmic Functions

SUMMARY POINTs

✓ Laws of logarithm:

1. $x = a^y \Leftrightarrow y = \log_a x$
2. $\log_a 1 = 0$
3. $\log_a a = 1$
4. $\log_a p + \log_a q = \log_a pq$
5. $\log_a p - \log_a q = \log_a \frac{p}{q}$
6. $\log_a p^n = n \log_a p$
7. $\log_b a = \frac{\log_c a}{\log_c b}$

✓ $f(x) = \frac{L}{1 + Ce^{-kx}}$: Logistic function, where L, C and k are positive constants

SUMMARY POINTs

✓ Semi-log model:

1. $y = k \cdot a^x \Leftrightarrow \ln y = (\ln a)x + \ln k$: Semi-log model
2. $\ln a$: Gradient of the straight line graph on $\ln y$-x plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-x plane

✓ Semi-log and log-log models:

1. $y = k \cdot x^n \Leftrightarrow \ln y = n \ln x + \ln k$: Log-log model
2. n : Gradient of the straight line graph on $\ln y$-$\ln x$ plane
3. $\ln k$: Vertical intercept of the straight line graph on $\ln y$-$\ln x$ plane

Solutions of Chapter 2

6 Paper 1 – Laws of Logarithm

Example

Find the value of each of the following, giving your answer as an integer.

(a) $\log 2 - \log 200$

[3]

(b) $\ln \frac{1}{e^{4.5}} + \ln \sqrt{e}$

[3]

Solution

(a) $\log 2 - \log 200 = \log \frac{2}{200}$ (A1) for correct formula

$\log 2 - \log 200 = \log \frac{1}{100}$

$\log 2 - \log 200 = \log 10^{-2}$ (A1) for valid approach

$\log 2 - \log 200 = -2$ A1

[3]

(b) $\ln \frac{1}{e^{4.5}} + \ln \sqrt{e} = \ln\left(\frac{1}{e^{4.5}} \cdot \sqrt{e}\right)$ (A1) for correct formula

$\ln \frac{1}{e^{4.5}} + \ln \sqrt{e} = \ln e^{-4.5+0.5}$

$\ln \frac{1}{e^{4.5}} + \ln \sqrt{e} = \ln e^{-4}$ (A1) for valid approach

$\ln \frac{1}{e^{4.5}} + \ln \sqrt{e} = -4$ A1

[3]

Exercise 6

1. Find the value of each of the following, giving your answer as an integer.

(a) $\log\frac{1}{25}+\log\frac{1}{40}$

[3]

(b) $\ln e^{7.5}-\ln e\sqrt{e}$

[3]

2. Find the value of each of the following, giving your answer as an integer.

(a) $\log 0.8+\log 1250$

[3]

(b) $\ln\sqrt[3]{e}-\ln e^{\frac{4}{3}}$

[3]

3. (a) Find the value of $\log 112+\log\frac{25}{4}-\log 7$.

[3]

(b) Solve the equation $e^{\ln\sqrt{x}}=3$.

[3]

4. (a) Find the value of $\ln\frac{1}{3}+\ln 45-\ln 15$.

[3]

(b) Solve the equation $10^{\log x^3}=1331$.

[3]

7 Paper 1 – Composite and Inverse Functions

2

Example

Let $f(x) = \log x^2$, for $x > 0$.

(a) Find $f^{-1}(x)$.

[3]

(b) Write down the range of f^{-1}.

[1]

Let $g(x) = \log x^3$, for $x > 0$.

(c) Find the value of $(f^{-1} \circ g)(100)$, giving your answer as an integer.

[4]

Solution

(a)

$y = \log x^2$	
$\Rightarrow x = \log y^2$	(M1) for swapping variables
$10^x = y^2$	
$\sqrt{10^x} = y$	
$(10^x)^{\frac{1}{2}} = y$	(A1) for changing subject
$\therefore f^{-1}(x) = 10^{\frac{1}{2}x}$	A1

[3]

(b)

$\{y : y > 0\}$	A1

[1]

(c)

$g(100) = \log 100^3$	(M1) for substitution
$g(100) = 6$	(A1) for correct value
$(f^{-1} \circ g)(100) = f^{-1}(g(100))$	
$(f^{-1} \circ g)(100) = f^{-1}(6)$	
$(f^{-1} \circ g)(100) = 10^{\frac{1}{2}(6)}$	(M1) for substitution
$(f^{-1} \circ g)(100) = 1000$	A1

[4]

Exercise 7

1. Let $f(x) = \log \sqrt[3]{x}$, for $x > 0$.

(a) Find $f^{-1}(x)$.

[3]

(b) Write down the range of f^{-1}.

[1]

Let $g(x) = \log x^4$, for $x > 0$.

(c) Find the value of $(f^{-1} \circ g)(\sqrt{10})$, giving your answer as an integer.

[4]

2. Let $f(x) = e^{4x}$.

(a) Find $f^{-1}(x)$.

[3]

(b) Write down the domain of f^{-1}.

[1]

Let $g(x) = (e^x - 1)^3$.

(c) Find the value of $(g \circ f^{-1})(16)$, giving your answer as an integer.

[4]

3. Let $f(x) = \ln x + 3$, for $x > 0$.

(a) Find $f^{-1}(x)$.

[3]

(b) Write down the range of f^{-1}.

[1]

Let $g(x) = e^{(x+1)(x-3)}$.

(c) Find the value of $(f \circ g)(2)$, giving your answer as an integer.

[4]

4. Let $f(x) = 10^{3x}$.

(a) Find $f^{-1}(x)$. [3]

(b) Write down the range of f^{-1}. [1]

Let $g(x) = (1 + \log x)^2$.

(c) Express $(g \circ f)(x)$ in the form $ax^2 + bx + c$, where a, b and c are integers. [3]

8 Paper 1 – Logistic Functions

Example

The population $P(t)$ of a town t years after 1st January, 2017 can be modelled by

$$P(t)=\frac{200}{1+4e^{-t}},\ t\geq 0.$$

(a) Find the population at the beginning of

(i) 2017;

(ii) 2021.

[4]

(b) Write down the range of $P(t)$.

[2]

Solution

(a) (i) The population at the beginning of 2017

$=P(0)$

$=\frac{200}{1+4e^{-0}}$ (M1) for substitution

$=40$ A1

(ii) The population at the beginning of 2021

$=P(4)$

$=\frac{200}{1+4e^{-4}}$ (M1) for substitution

$=186.3476919$

$=186$ A1

[4]

(b) $\{P:40\leq P<200\}$ A2

[2]

Exercise 8

1. In an experiment, the amount of bacteria $B(t)$ (in million) in a culture t hours after 1 a.m. on the first day can be modelled by $B(t)=\dfrac{12}{1+5e^{-0.06t}}$, $t\geq 0$.

(a) Find the amount of bacteria at

(i) 1 a.m. on the first day;

(ii) 3 a.m. on the second day.

[4]

(b) Write down the range of $B(t)$.

[2]

2. The weight of a substance $W(t)$ (in gram) t days after 1st July can be modelled by $W(t)=\dfrac{27}{1+8e^{-0.11t}}$, $t\geq 0$.

(a) Find the weight of the substance on 1st July.

[2]

(b) Find the date that the weight of the substance first reaches 20g.

[2]

(c) Write down the horizontal asymptote of the graph of $W(t)$.

[1]

3. The population $P(t)$ of a city t years after 1st January, 2010 can be modelled by $P(t)=\dfrac{600000}{1+Ce^{-0.11t}}$, where $C\in\mathbb{R}$, $t\geq 0$. It is given that $P(0)=100000$.

(a) Find C.

[2]

(b) Hence, find t when $P(t)=450000$.

[2]

(c) Write down the horizontal asymptote of the graph of $P(t)$.

[1]

4. The value $V(t)$ of a vase (in dollars) t years after 31st December, 1900 can be modelled by $V(t)=\begin{cases}\dfrac{250000}{1+19e^{-1.6}}(t+10) & 0\le t<10\\ \dfrac{5000000}{1+19e^{-0.16t}} & t\ge 10\end{cases}$.

(a) Find the value of the vase at the end of 1907.

[2]

(b) Find t when $V(t)=2000000$.

[2]

(c) Interpret the condition on the value of the vase after a long period of time.

[1]

9

Paper 1 – Semi-Log and Log-Log Models

Example

The relationship between the air pressure $P(t)$ (in Pa) in an instrument, $P(t) > 10$, and the time t minutes after the start of the experiment is given by $P(t) = 10 + he^{kt}$, where h, $k \in \mathbb{R}$.

(a) Express $\ln(P-10)$ in the form $at+b$.

[3]

A graph of $\ln(P-10)$ against t shows a straight line such that the gradient and the vertical intercept of the line are 0.015 and 2.8 respectively.

(b) Write down the value of k.

[1]

(c) Find the value of h, correct the answer to 4 decimal places.

[2]

Solution

(a)

$P = 10 + he^{kt}$	
$P - 10 = he^{kt}$	
$\ln(P-10) = \ln(he^{kt})$	(A1) for correct approach
$\ln(P-10) = \ln h + \ln e^{kt}$	(A1) for correct approach
$\ln(P-10) = kt + \ln h$	A1

[3]

(b)

0.015	A1

[1]

(c)

$\ln h = 2.8$	
$h = e^{2.8}$	(M1) for valid approach
$h = 16.44464677$	
$h = 16.4446$	A1

[2]

Exercise 9

1. The relationship between two variables T and x is given by $T = hk^x$, where h, $k \in \mathbb{R}$, $T > 0$.

(a) Express $\ln T$ in terms of x.

[3]

A graph of $\ln T$ against x shows a straight line such that the gradient and the vertical intercept of the line are 0.06 and -1.7 respectively.

(b) Find, correct the answers to 4 decimal places, the value of

(i) h;

(ii) k.

[4]

2. The relationship between two variables W and x is given by $W = hx^k$, where h, $k \in \mathbb{R}$, W, $x > 0$.

(a) Express $\ln W$ in terms of $\ln x$.

[3]

A graph of $\ln W$ against $\ln x$ shows a straight line such that it passes through the points $(0.3, 0)$ and $(0, 1.5)$.

(b) Find the value of k.

[2]

(c) Find the value of h, correct the answer to 4 decimal places.

[2]

3. The relationship between two variables V and x can be modelled by $\ln V = 3x + b$, where $b \in \mathbb{R}$, $V > 0$. A graph of $\ln V$ against x shows a straight line such that the gradient and the horizontal intercept of the line are 3 and -6 respectively.

(a) Find the value of b.

[2]

(b) Hence, express V in terms of x.

[1]

(c) Find the ratio of the values of V when x increases from 0.5 to 1.

[3]

4. The relationship between two variables N and t can be modelled by $\ln N = a\ln t + b$, where a, $b \in \mathbb{R}$, N, $t > 0$. A graph of $\ln N$ against $\ln t$ shows a straight line such that it passes through the points $(2.5, 1)$ and $(5, 0)$.

(a) Find the values of

(i) a;

(ii) b.

[4]

(b) Hence, express N in terms of t.

[3]

10 Paper 2 – Comparing Two Models

Example

The number of insects in two colonies, A and B, starts increasing at the same time.

The number of insects in colony A after t months is modeled by the function $A(t) = 240e^{0.3t}$.

(a) Find the initial number of insects in colony A.

[2]

(b) Find the number of insects in colony A after seven months.

[2]

(c) How long does it take for the number of insects in colony A to reach 800?

[3]

The number of insects in colony B after t months is modeled by the function $B(t) = 360e^{kt}$.

(d) After ten months, there are 1000 insects in colony B. Find the value of k.

[3]

The number of insects in colony A first exceeds the number of insects in colony B after n months, where $n \in \mathbb{Z}$.

(e) Find the value of n.

[4]

Solution

(a) Initial number of insects

$= 240e^{0.3(0)}$ (A1) for substitution

$= 240$ A1

[2]

(b) Number of insects in colony A after seven months

$= 240e^{0.3(7)}$ (A1) for substitution

$= 1959.880779$

$= 1960$ A1

[2]

(c) $A(t)=800$ (M1) for setting equation

$240e^{0.3t}=800$

$240e^{0.3t}-800=0$ (A1) for correct approach

By considering the graph of $y=240e^{0.3t}-800$,

$t=4.0132427$.

$\therefore$ It takes 4.01 months. A1

[3]

(d) $B(10)=1000$ (M1) for setting equation

$360e^{10k}=1000$

$360e^{10k}-1000=0$ (A1) for correct approach

By considering the graph of $y=360e^{10k}-1000$,

$k=0.1021651$.

$\therefore k=0.102$ A1

[3]

(e) $A(t)>B(t)$ (M1) for setting inequality

$A(t)-B(t)>0$

$240e^{0.3t}-360e^{0.1021651t}>0$ A1

By considering the graph of

$y=240e^{0.3t}-360e^{0.1021651t}$, $t>2.0495125$. (A1) for correct approach

$\therefore n=3$ A1

[4]

Exercise 10

1. The number of leopards and tigers in a forest start increasing at the same time.

The number of leopards in the forest after t years is modeled by the function $A(t)=2500e^{0.075t}$.

(a) Find the initial number of leopards.

[2]

(b) Find the number of leopards after ten years.

[2]

(c) How long does it take for the number of leopards to reach 8000?

[3]

The number of tigers in the forest after t years is modeled by the function $B(t)=ke^{\frac{180}{k}t}$, where $k<2000$.

(d) After ten years, there are 5000 tigers. Find the value of k.

[3]

(e) The number of tigers first exceeds the number of leopards after n years, where $n\in\mathbb{Z}$. Find the value of n.

[4]

2. The number of trams and the number of people using trams in a city is studied.

The number of trams in the city after t years is modeled by the function $A(t)=420\times1.15^{t}$.

(a) Find the initial number of trams.

[2]

(b) Find the number of trams after six years.

[2]

(c) How long does it take for the number of trams to reach 750?

[3]

The number of people using trams in the city after t years is modeled by the function $B(t)=\dfrac{4680000}{70e^{-kt}+130}$.

(d) After five years, there are 27500 people using trams. Find the value of k.

[4]

(e) The number of trams first exceeds five times the number of people using trams after n years, where $n\in\mathbb{Z}$. Find the value of n.

[4]

3. The number of food delivery cars and the number of people using food delivery cars in a town is studied.

The number of food delivery cars in the town after t weeks is modeled by the function $A(t) = 1050 \times 1.25^t$.

(a) Find the initial number of food delivery cars.

[2]

(b) Find the number of food delivery cars after sixteen weeks.

[2]

(c) How long does it take for the number of food delivery cars to reach 4200?

[4]

The number of people using food delivery cars in the town after t weeks is modeled by the function $B(t) = \dfrac{410000}{75k + 95e^{-kt}}$.

(d) After twelve weeks, there are 4600 people using food delivery cars. Find the value of k.

[3]

(e) The number of food delivery cars first exceeds double the number of people using food delivery cars after n weeks, where $n \in \mathbb{Z}$. Find the value of n.

[4]

4. The air pressure in two machines, A and B, are recorded in an experiment.

The air pressure in machine A after t minutes is modeled by the function $P(t) = 4e^{0.12t}$.

(a) Find the air pressure in machine A after half an hour.

[2]

(b) How long does it take for the air pressure in machine A to reach 8 units?

[3]

The air pressure in machine B after t minutes is modeled by the function $Q(t) = Q_0 e^{kt}$. It is recorded that the initial air pressure and the air pressure in machine B after half an hour are 3.5 units and 171 units respectively.

(c) Find the values of

(i) Q_0;

(ii) k.

[5]

(d) The sum of air pressures in two machines first exceeds 400 units after n minutes, where $n \in \mathbb{Z}$. Find the value of n.

[4]

Chapter

3

Geometric Sequences

SUMMARY POINTs

✓ $S_\infty = \frac{u_1}{1-r}$: The sum to infinity of a geometric sequence u_n, given that $-1 < r < 1$

Solutions of Chapter 3

11 Paper 1 – Sum to Infinity

Example

The first three terms of a geometric sequence are $u_1 = 800$, $u_2 = 720$ and $u_3 = 648$.

(a) Find the value of r.

[2]

(b) Find the value of S_6.

[2]

(c) Find the sum to infinity of this sequence.

[2]

Solution

(a) $r = \frac{720}{800}$ (M1) for valid approach

$r = 0.9$ A1

[2]

(b) $S_6 = \frac{u_1(1-r^6)}{1-r}$

$S_6 = \frac{800(1-(0.9)^6)}{1-0.9}$ (A1) for substitution

$S_6 = 3748.472$

$S_6 = 3750$ A1

[2]

(c) $S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{800}{1-0.9}$ (A1) for substitution

$S_\infty = 8000$ A1

[2]

Exercise 11

1. The first three terms of a geometric sequence are $u_1 = -900$, $u_2 = -540$ and $u_3 = -324$.

(a) Find the value of r.

[2]

(b) Find the value of S_{10}.

[2]

(c) Find the sum to infinity of this sequence.

[2]

2. The first three terms of an infinite geometric sequence are $\ln x^{48}$, $\ln x^{24}$ and $\ln x^{12}$, where $x > 0$.

(a) Find the common ratio of the geometric sequence.

[3]

(b) Find u_6.

[2]

(c) Find the sum to infinity of this sequence.

[2]

3. The first three terms of an infinite geometric sequence are e^{12x}, e^{8x} and e^{4x}.

(a) Find the common ratio of the geometric sequence.

[2]

(b) Find u_7.

[2]

(c) Find x if the sum to infinity of this sequence is $\dfrac{e^{96}}{e^{24}-1}$.

[3]

4. The first three terms of an infinite geometric sequence are 3^{10x}, 3^{9x} and 3^{8x}.

(a) Find the common ratio of the geometric sequence.

[2]

(b) Find a general expression for u_n.

[3]

(c) Find the sum to infinity if the common ratio is $\dfrac{1}{3}$, giving the answer in the form $a \times 3^b$.

[3]

12 Paper 1 – Condition of Sum to Infinity

Example

The first three terms of an infinite geometric sequence are $-\frac{6}{r}$, -6, $-6r$, where r is the common ratio. The two possible values of r are $\frac{3}{2}$ and $-\frac{2}{3}$.

(a) State which value of r leads to this sum **and** justify your answer.

[2]

(b) Hence, calculate the sum of the sequence.

[4]

Solution

(a) $r = -\frac{2}{3}$ leads to a finite sum. A1

As $-1 < -\frac{2}{3} < 1$. R1

[2]

(b) $u_1 = -\frac{6}{-\frac{2}{3}}$ (A1) for finding u_1

$u_1 = 9$ (A1) for correct value

$S_\infty = \frac{u_1}{1-r}$

$S_\infty = \frac{9}{1-\left(-\frac{2}{3}\right)}$ (A1) for substitution

$S_\infty = \frac{27}{5}$ A1

[4]

Exercise 12

1. The first three terms of an infinite geometric sequence are $\frac{10}{r}$, 10, $10r$, where r is the common ratio. The two possible values of r are $\frac{1}{2}$ and -2.

(a) State which value of r leads to this sum **and** justify your answer.

[2]

(b) Hence, calculate the sum of the sequence.

[4]

2. The first three terms of an infinite geometric sequence are $\frac{27}{r^2}$, $\frac{27}{r}$, 27, where r is the common ratio. The two possible values of r are 3 and $-\frac{1}{3}$.

(a) If the sequence has a finite sum, state which value of r leads to this sum **and** justify your answer.

[2]

(b) If the sequence does not have a finite sum, find the sum of the first four terms.

[4]

3. The first three terms of an infinite geometric sequence are $\log_2 x^r$, $\log_2 x^{r^2}$, $\log_2 x^{r^3}$, where r is the common ratio. The two possible values of the common ratio r are $\frac{1}{2}$ and -2.

(a) Consider the value of r such that $-1 < r < 1$. Find S_∞, giving the answer in terms of x.

[3]

(b) Consider the value of r such that $r < -1$. Find S_6 when $x = \frac{1}{2}$.

[4]

4. The first three terms of an infinite geometric sequence are u_1, $u_2 = m + 2$, $u_3 = 9$, where $m \in \mathbb{Z}$. The two possible values of the common ratio r are $\frac{4}{3}$ and $-\frac{1}{3}$.

(a) Consider the value of r such that $-1 < r < 1$. Find m.

[3]

(b) Hence, calculate the sum of the sequence.

[4]

Chapter

4

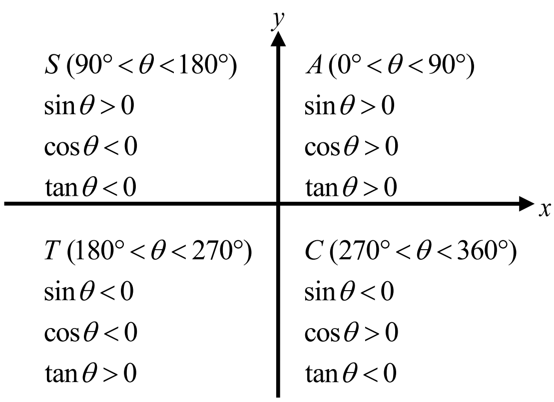
Trigonometry

SUMMARY POINTs

✓ Trigonometric identities:

1. $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$
2. $\sin^2\theta + \cos^2\theta \equiv 1$

✓ ASTC diagram



Solutions of Chapter 4

13 Paper 1 – Trigonometric Identities

Example

Let $\cos\theta = 0.28$, where θ is acute. Using trigonometric identities to find

(a) $\sin\theta$;

[2]

(b) $\tan\theta$.

[2]

Solution

(a) $\sin\theta = \sqrt{1-\cos^2\theta}$

$\sin\theta = \sqrt{1-0.28^2}$ (A1) for substitution

$\sin\theta = 0.96$ A1

[2]

(b) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

$\tan\theta = \dfrac{0.96}{0.28}$ (A1) for substitution

$\tan\theta = \dfrac{24}{7}$ A1

[2]

Exercise 13

1. Let $\sin\theta = \dfrac{5}{13}$, where θ is obtuse. Using trigonometric identities to find

(a) $\cos\theta$;

[3]

(b) $\tan\theta$.

[2]

2. Let $\tan\theta = -\frac{\sqrt{11}}{5}$, where $\frac{3\pi}{2} < \theta < 2\pi$.

(a) Show that $\cos\theta = \frac{5}{6}$.

[2]

(b) Using trigonometric identities to find $\sin\theta$.

[3]

3. Let $625\cos^4\theta = 256$, where $\pi < \theta < \frac{3\pi}{2}$.

(a) (i) Write down the value of $\cos^2\theta$.

(ii) Hence, find $\sin\theta$.

[4]

(b) Find $\tan\theta$.

[2]

4. Let $\tan^2\theta = \frac{81}{1600}$, where θ is obtuse.

(a) (i) Write down the exact value of $\tan\theta$.

(ii) Hence, find $\sin\theta$.

[3]

(b) Find $\cos\theta$.

[3]

14 Paper 1 – Solving Trigonometric Equations

Example

(a) Solve $\cos 2x = -3\sin x - 4$ for $0 \le x \le 6\pi$.

[4]

(b) Write down the number of solution(s) of $1 + \cos 2x = -3\sin x - 4$ for $0 \le x \le 6\pi$.

[1]

Solution

(a) $\cos 2x = -3\sin x - 4$

$\cos 2x + 3\sin x + 4 = 0$ (M1) for valid approach

By considering the graph of $y = \cos 2x + 3\sin x + 4$,

$x = 4.7123885$, $x = 10.995574$ or $x = 17.278759$.

$\therefore x = 4.71$, $x = 11.0$ or $x = 17.3$ A3

[4]

(b) 0 A1

[1]

Exercise 14

1. (a) Solve $5 - 2\cos^2 x = 3\cos x$ for $0 \le x \le 4\pi$.

[4]

(b) Write down the number of solution(s) of $\pi - 2\cos^2 x = 3\cos x$ for $0 \le x \le 4\pi$.

[1]

2. (a) Solve $\cos 2x = 4 - 7\sin x$ for $0 \le x \le 2\pi$.

[3]

(b) Write down the number of solution(s) of $\cos 2x - 2 = 4 - 7\sin x$ for $0 \le x \le 2\pi$.

[1]

3. (a) Solve $\sin x - \sin x \cos x = 0$, for $0 < x < 4\pi$.

[3]

(b) Write down the number of solution(s) of $\sin x - \sin x \cos x = 0$ for $-5\pi \le x \le \pi$.

[1]

4. (a) Solve $\cos 2x - \sin 4x = 0$, for $0 \le x \le \frac{\pi}{2}$.

[3]

(b) Write down the number of solution(s) of $\cos 2x - \sin 4x = 0$ for $-2\pi \le x \le -\pi$.

[1]

15 Paper 1 – Composite Trigonometric Functions

Example

Let $f(x)=\frac{2}{3}x-1$, $g(x)=5\cos\left(\frac{x}{2}\right)+2$. Let $h(x)=(g\circ f)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the period of h.

[2]

(c) Write down the range of h.

[2]

Solution

(a) $h(x)=g(f(x))$ (M1) for composite function

$h(x)=5\cos\left(\frac{f(x)}{2}\right)+2$ (A1) for substitution

$h(x)=5\cos\left(\frac{1}{3}x-\frac{1}{2}\right)+2$ A1

[3]

(b) The period of h

$=2\pi\div\frac{1}{3}$ (M1) for valid approach

$=6\pi$ A1

[2]

(c) $\{y:-3\le y\le 7\}$. A2

[2]

Exercise 15

1. Let $f(x)=2x+3$, $g(x)=3\cos\left(\frac{x}{4}\right)-5$. Let $h(x)=(g\circ f)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the period of h.

[2]

(c) Write down the range of h.

[2]

2. Let $f(x)=8x+7$, $g(x)=4\sin\left(\frac{x}{2}\right)-3$. Let $h(x)=(g\circ f)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the period of h.

[2]

(c) Write down the amplitude of h.

[1]

3. Let $f(x)=\frac{3}{2}x-1$, $g(x)=4\sin\left(\frac{x}{3}\right)+13$. Let $h(x)=(f\circ g)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the amplitude of h.

[1]

(c) Write down the range of h.

[2]

4. Let $f(x)=1-2x$, $g(x)=6\cos\pi x+1$. Let $h(x)=(f\circ g)(x)$.

(a) Find an expression for $h(x)$.

[3]

(b) Find the amplitude of h.

[1]

(c) Write down the range of h.

[2]

Chapter

5

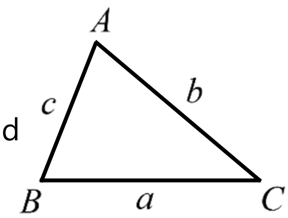
2-D Trigonometry

SUMMARY POINTs

✓ Consider a triangle ABC:

$\frac{\sin A}{a} = \frac{\sin B}{b}$ or $\frac{a}{\sin A} = \frac{b}{\sin B}$: Sine rule

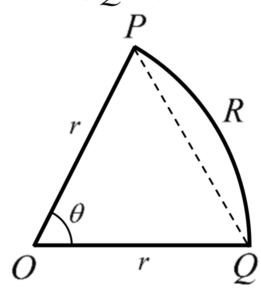
Note: The ambiguous case exists if two sides and an angle are known, and the angle is opposite to the shorter known side



✓ $\frac{x^\circ}{180^\circ} = \frac{y \text{ rad}}{\pi \text{ rad}}$: Method of conversions between degree and radian

✓ Consider a sector $OPRQ$ with centre O, radius r and $\angle POQ = \theta$ in radian:

1. $r\theta$: Arc length PQ
2. $\frac{1}{2}r^2\theta$: Area of the sector $OPRQ$
3. $\frac{1}{2}r^2(\theta - \sin\theta)$: Area of the segment PRQ

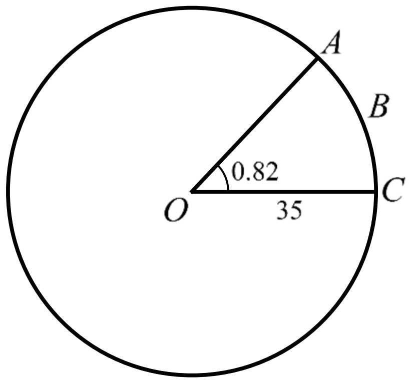


Solutions of Chapter 5

16 Paper 1 – Areas and Perimeters of Sectors

Example

The following diagram shows a circle with centre O and radius 35 cm.



The points A, B and C are on the circumference of the circle and $A\hat{O}C = 0.82$ rad.

(a) (i) Find the length of arc ABC.

(ii) Hence, find the perimeter of sector $OABC$.

[4]

(b) Find the exact area of sector $OABC$.

[2]

Solution

(a) (i) The length of arc ABC

$= (35)(0.82)$ (A1) for substitution

$= 28.7$ cm A1

(ii) The perimeter of sector $OABC$

$= 28.7 + 35 + 35$ (M1) for valid approach

$= 98.7$ cm A1

[4]

(b) The area of sector $OABC$

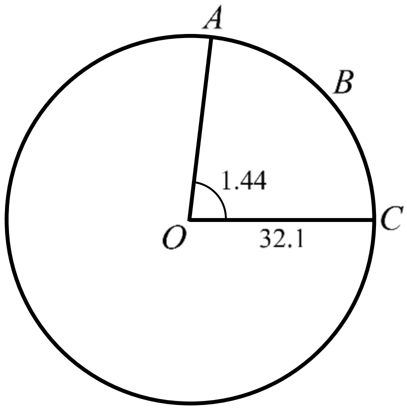
$= \frac{1}{2}(35)^2(0.82)$ (A1) for substitution

$= 502.25\text{ cm}^2$ A1

[2]

Exercise 16

1. The following diagram shows a circle with centre O and radius 32.1 cm.



The points A, B and C are on the circumference of the circle and $A\hat{O}C = 1.44$ rad.

(a) Find the length of arc ABC.

[2]

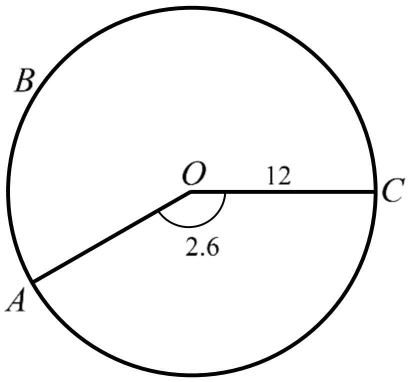
(b) Find the perimeter of sector $OABC$.

[2]

(c) Find the area of sector $OABC$.

[2]

2. The following diagram shows a circle with centre O and radius 12 cm.



The points A, B and C are on the circumference of the circle and $A\hat{O}C = 2.6$ rad.

(a) Find the length of arc ABC.

[3]

(b) Find the perimeter of sector $OABC$.

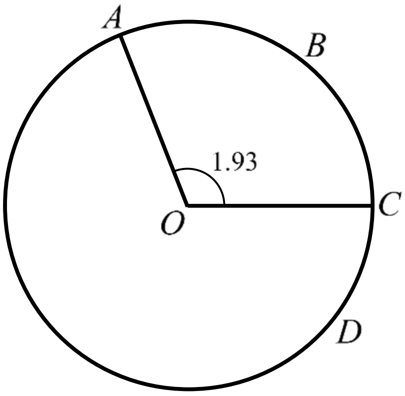
[2]

(c) Find the area of sector $OABC$.

[2]

5

3. The following diagram shows a circle with centre O.



The points A, B, C and D are on the circumference of the circle such that $A\hat{O}C = 1.93\text{ rad}$ and the area of the sector $OABC$ is 603 cm^2.

(a) Find the length of OC.

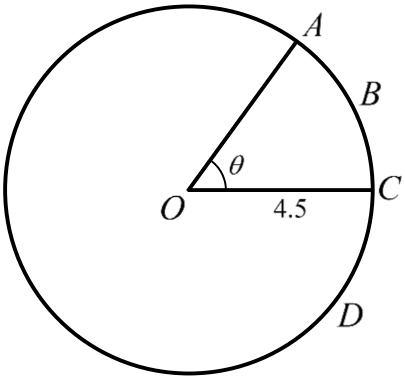
[2]

(b) (i) Find the reflex $A\hat{O}C$.

(ii) Hence, find the area of sector $OADC$.

[4]

4. The following diagram shows a circle with centre O and radius 4.5 cm.



The points A, B, C and D are on the circumference of the circle such that the length of the arc ABC is 4.32 cm.

(a) Find the value of θ, giving the answer in radians.

[2]

(b) (i) Find the reflex $A\hat{O}C$.

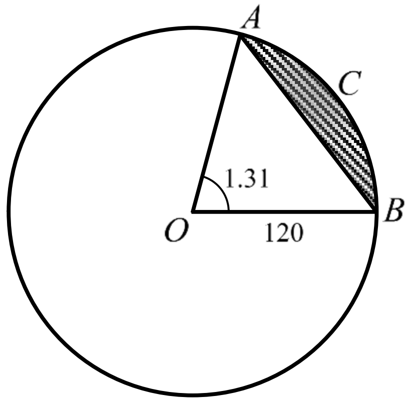
(ii) Hence, find the area of sector $OADC$.

[4]

17 Paper 1 – Areas and Perimeters of Segments

Example

The following diagram shows a circle with centre O and radius 120 cm.



The points A, B and C are on the circumference of the circle, and $A\hat{O}B = 1.31$ rad.

(a) Find the exact length of arc ACB. [2]

(b) Find the length of AB. [3]

(c) Hence, find the perimeter of the shaded segment ABC. [2]

Solution

(a) The exact length of arc ACB

$= (120)(1.31)$ (A1) for substitution

$= 157.2$ cm A1

[2]

(b) $AB = \sqrt{120^2 + 120^2 - 2(120)(120)\cos 1.31}$ (M1)(A1) for substitution

$AB = 146.1982184$

$AB = 146$ cm A1

[3]

(c) The required perimeter

$= 146.1982184 + 157.2$ (M1) for correct approach

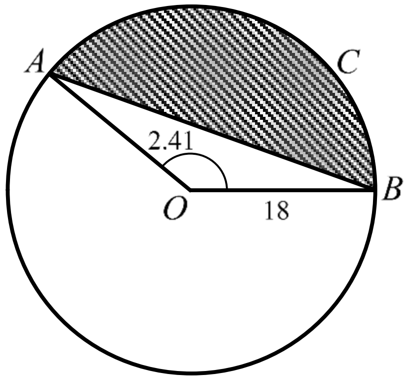
$= 303.3982184$

$= 303$ cm A1

[2]

Exercise 17

1. The following diagram shows a circle with centre O and radius 18 cm.



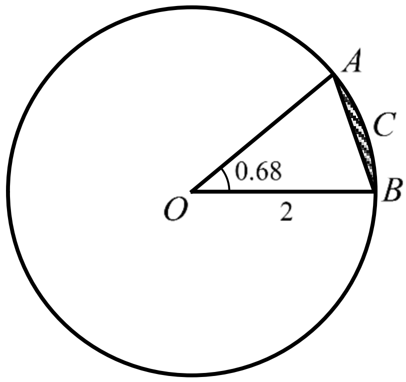
The points A, B and C are on the circumference of the circle, and $A\hat{O}B = 2.41$ rad.

(a) Find the exact length of arc ACB. [2]

(b) Find the length of AB. [3]

(c) Hence, find the perimeter of the shaded segment ABC. [2]

2. The following diagram shows a circle with centre O and radius 2 cm.



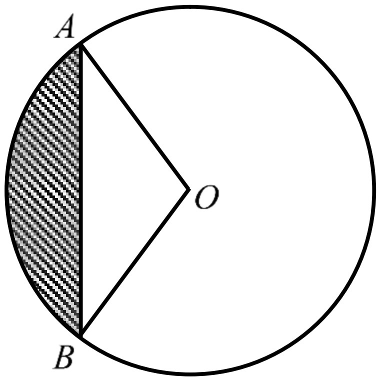
The points A, B and C are on the circumference of the circle, and $A\hat{O}B = 0.68$ rad.

(a) Find the area of the sector $OACB$. [2]

(b) Find the area of the triangle OAB. [2]

(c) Hence, find the area of the shaded segment ABC. [2]

3. The following diagram shows the chord AB in a circle of radius 15.6 cm, where $AB = 25\,cm$.



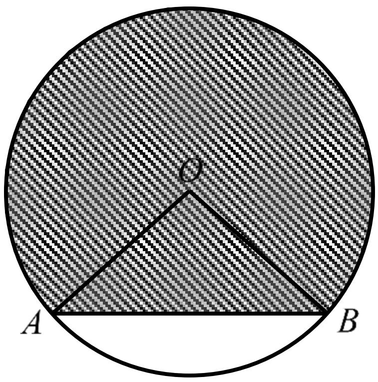
(a) Find the size of the minor angle $A\hat{O}B$, giving the answer in radians. [3]

(b) Find the area of the sector AOB. [2]

It is given that the area of the triangle AOB is $117\ cm^2$.

(c) Find the area of the shaded region. [2]

4. The following diagram shows the chord AB in a circle of radius 24 cm, where $AB = 37\,cm$.



(a) Find the size of the minor angle $A\hat{O}B$, giving the answer in radians. [3]

(b) Find the length of the minor arc AB. [2]

(c) Hence, find the perimeter of the shaded segment. [3]

5

18 Paper 1 – Ambiguous Case in Sine Rule

Example

Consider the triangle ABC such that $BC = 12$, $AC = 9$ and $A\hat{B}C = 28^\circ$.

(a) Find the possible values of $B\hat{A}C$.

[4]

(b) If $B\hat{A}C$ is an acute angle, find AB.

[2]

Solution

(a)
$$\frac{\sin B\hat{A}C}{BC} = \frac{\sin A\hat{B}C}{AC}$$ (M1) for sine rule

$$\frac{\sin B\hat{A}C}{12} = \frac{\sin 28^\circ}{9}$$ (A1) for substitution

$$\sin B\hat{A}C = \frac{12 \sin 28^\circ}{9}$$

$$B\hat{A}C = 38.75283673^\circ \text{ or } 180^\circ - 38.75283673^\circ$$

$$B\hat{A}C = 38.8^\circ \text{ or } 141^\circ$$ A2

[4]

(b)
$$\frac{AB}{\sin A\hat{C}B} = \frac{AC}{\sin A\hat{B}C}$$

$$\frac{AB}{\sin(180^\circ - 28^\circ - 38.75283673^\circ)} = \frac{9}{\sin 28^\circ}$$ (A1) for substitution

$$AB = 17.61405255$$

$$AB = 17.6$$ A1

[2]

Exercise 18

1. Consider the triangle ABC such that $B\hat{A}C = 20^\circ$, $AC = 80$ and $BC = 30$.

(a) Find the possible values of $A\hat{B}C$.

[4]

(b) If $A\hat{B}C$ is an obtuse angle, find $A\hat{C}B$.

[2]

2. Consider the triangle ABC such that $BC = 20$, $AC = \sqrt{151}$ and $A\hat{B}C = 33^\circ$.

5

(a) Find the possible values of $B\hat{A}C$.

[4]

(b) Hence, find the possible values of AB.

[3]

(c) For the above two cases, write down the difference between the perimeters of the triangle ABC.

[1]

3. Consider the triangle ABC such that $B\hat{A}C = 18^\circ$, $AC = \sqrt{35}$ and $BC = 2.7$.

(a) Find the possible values of $A\hat{B}C$.

[4]

(b) Hence, find the area of the triangle if it is greater than 6.

[3]

4. Consider the triangle ABC such that $BC = 37$, $AC = 27$ and $A\hat{B}C = 41^\circ$.

(a) Find the possible values of $B\hat{A}C$.

[4]

It is given that the perimeter of the triangle ABC is less than 84.

(b) (i) Write down an inequality of AB.

(ii) Hence, find the perimeter of the triangle ABC.

[4]

Chapter

6

Complex Numbers

SUMMARY POINTs

✓ Terminologies of complex numbers:

$\mathrm{i}=\sqrt{-1}$: Imaginary unit

$z=a+b\mathrm{i}$: Complex number in Cartesian form

a : Real part of z

b : Imaginary part of z

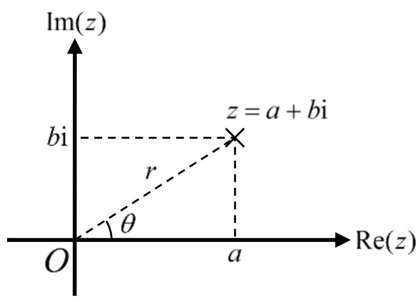
$z^*=a-b\mathrm{i}$: Conjugate of $z=a+b\mathrm{i}$

$|z|=r=\sqrt{a^2+b^2}$: Modulus of $z=a+b\mathrm{i}$

$\arg(z)=\theta=\arctan\frac{b}{a}$: Argument of $z=a+b\mathrm{i}$

✓ Properties of Argand diagram:

1. Real axis: Horizontal axis
2. Imaginary axis: Vertical axis



SUMMARY POINTs

✓ Forms of complex numbers:

1. $z = a + b\mathrm{i}$: Cartesian form
2. $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$: Modulus-argument form
3. $z = re^{\mathrm{i}\theta}$: Euler form

✓ Properties of moduli and arguments of complex numbers z_1 and z_2 :

1. $|z_1 z_2| = |z_1||z_2|$
2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
3. $\arg(z_1 z_2) = \arg z_1 + \arg z_2$
4. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

✓ If $z = a + b\mathrm{i}$ is a root of the polynomial equation $p(z) = 0$, then $z^* = a - b\mathrm{i}$ is also a root of $p(z) = 0$

6

Solutions of Chapter 6

19 Paper 1 – Real and Imaginary Parts

Example

(a) Express $z=\frac{1}{2+\text{i}}$ in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[2]

(b) Express z^3 in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[2]

(c) Hence, write down the imaginary part of z^3.

[1]

Solution

(a) $\frac{1}{2+\text{i}}=\frac{2}{5}-\frac{1}{5}\text{i}$ A2

[2]

(b) $z^3=\left(\frac{2}{5}-\frac{1}{5}\text{i}\right)^3$

$z^3=\frac{2}{125}-\frac{11}{125}\text{i}$ A2

[2]

(c) $-\frac{11}{125}$ A1

[1]

Exercise 19

1. (a) Express $z=\frac{1}{3-4\text{i}}$ in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[2]

(b) Express z^2 in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[2]

(c) Hence, write down the real part of z^2.

[1]

2. z is a complex number such that $\dfrac{z}{1-z}=-1-0.5\text{i}$.

(a) Express z in the form $a+b\text{i}$, where a, $b\in\mathbb{Z}$.

[3]

(b) Hence, write down the imaginary part of z.

[1]

3. z is a complex number such that $2z-1-\text{i}=5+7\text{i}$.

(a) Express z in the form $a+b\text{i}$, where a, $b\in\mathbb{Z}$.

[2]

(b) Express z^4 in the form $a+b\text{i}$, where a, $b\in\mathbb{Z}$.

[2]

(c) Hence, write down the real part of z^4.

[1]

6

4. z is a complex number such that $\dfrac{z}{5-12\text{i}}=\dfrac{24-7\text{i}}{\text{i}}$.

(a) Express z in the form $a+b\text{i}$, where a, $b\in\mathbb{Z}$.

[2]

(b) Express $(\text{i}^3z)^2$ in the form $a+b\text{i}$, where a, $b\in\mathbb{Z}$.

[2]

(c) Hence, write down the imaginary part of $(\text{i}^3z)^2$.

[1]

20 Paper 1 – Moduli and Arguments

Example

Consider the complex number $z = \frac{5i}{3+4i}$, where $z \in \mathbb{C}$.

(a) Express z in the form $a + ib$, where a, $b \in \mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z.

[2]

(c) Find the value of the argument of z.

[2]

Solution

(a)

$z = \frac{5i}{3+4i}$

$z = \frac{4}{5} + \frac{3}{5}i$ A2

[2]

(b) The modulus of z

$= \sqrt{\left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2}$ M1

$= \sqrt{1}$

$= 1$ A1

[2]

(c) The argument of z

$= \arctan\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)$ M1

$= \arctan\left(\frac{3}{4}\right)$

$= 0.6435011088$ rad

$= 0.644$ rad A1

[2]

Exercise 20

1. Consider the complex number $z=\frac{2-\mathrm{i}}{2+\mathrm{i}}$, where $z \in \mathbb{C}$.

(a) Express z in the form $a+\mathrm{i}b$, where a, $b \in \mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z.

[2]

(c) Find the value of the argument of z.

[2]

2. Consider the complex number $z=\frac{10}{13}-\frac{24}{13}\mathrm{i}$, where $z \in \mathbb{C}$.

(a) Express z^2 in the form $a+\mathrm{i}b$, where a, $b \in \mathbb{Q}$.

[2]

(b) Find the exact value of the modulus of z^2.

[2]

(c) Find the value of the argument of z^2.

[2]

3. Consider the complex number $z=\frac{8}{5}-\frac{6}{5}\mathrm{i}$, where $z \in \mathbb{C}$.

(a) (i) Express z^3 in the form $a+\mathrm{i}b$, where a, $b \in \mathbb{Q}$.

(ii) Hence, write down $(z^3)^*+\frac{352}{125}$ in the form $a+\mathrm{i}b$, where a, $b \in \mathbb{Q}$.

[3]

(b) Find the exact value of the modulus of $(z^3)^*+\frac{352}{125}$.

[2]

(c) Write down the exact value of the argument of $(z^3)^*+\frac{352}{125}$.

[1]

4. The complex numbers z_1 and z_2 have arguments between 0 and π radians. Given that $z_1+\mathrm{i}z_2=0$, $z_2{}^2=-2-2\sqrt{a}\mathrm{i}$ and $|z_2|=2$, where $a \in \mathbb{R}$.

(a) Find the modulus of z_1.

[2]

(b) Find the value of a.

[3]

(c) Hence, find the argument of $z_2{}^2$.

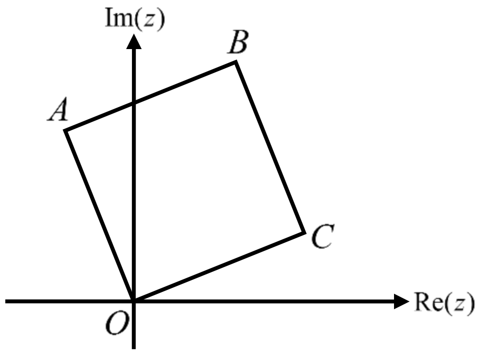
[2]

6

21 Paper 1 – Argand Diagrams

Example

In the following Argand diagram with the origin O, the point A represents the complex number $-4+10i$. The shape of $OABC$ is a square.



(a) Determine the complex numbers represented by

(i) the point B;

(ii) the point C.

[4]

(b) Hence, find the area of $OABC$.

[2]

Solution

(a) (i) $z_B = -4+10i+(10+4i)$ M1

$z_B = 6+14i$ A1

$z_C = 0+0i+(10+4i)$ M1

$z_C = 10+4i$ A1

[4]

(b) The area of $OABC$

$=(OA)^2$ M1

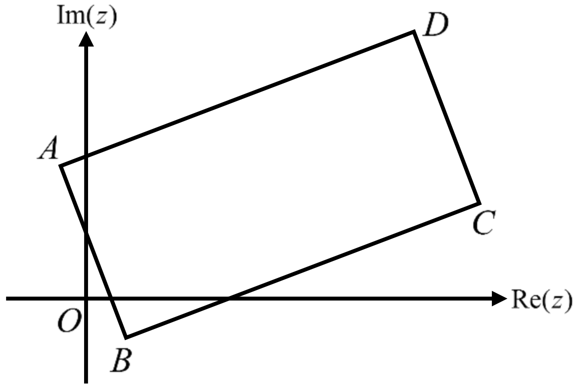
$=(\sqrt{(-4)^2+10^2})^2$

$=116$ A1

[2]

Exercise 21

1. In the following Argand diagram with the origin O, the point A and the point B represent the complex numbers $-2+9i$ and $3-3i$ respectively. The shape of $ABCD$ is a rectangle such that $AD = 2AB$.



(a) Write down

(i) $\text{Re}(3-3i) - \text{Re}(-2+9i)$;

(ii) $\text{Im}(3-3i) - \text{Im}(-2+9i)$.

[2]

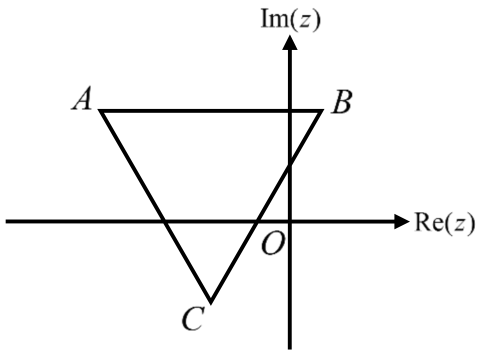
(b) Find the length of AB.

[2]

(c) Hence, find the area of $ABCD$.

[2]

2. In the following Argand diagram with the origin O, the point A represents the complex numbers $-18+10i$. The shape of ABC is an equilateral triangle with the horizontal side $AB = 20$.



(a) Determine the complex numbers represented by

(i) the point B;

(ii) the point C, giving the answer in exact value.

[4]

(b) Find the area of ABC.

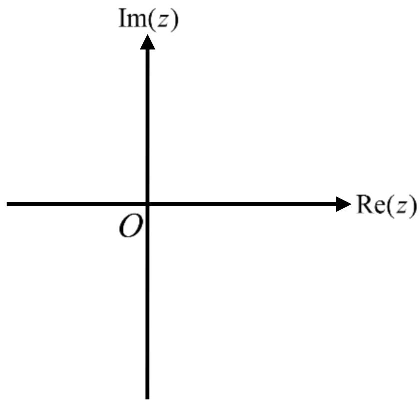
[2]

6

3. In an Argand diagram with the origin O, the points A, B, C and D represent the complex numbers $z=4+4\text{i}$, z^*, $\omega=-2+2\text{i}$ and ω^* respectively.

(a) Sketch the points A, B, C and D on the following Argand diagram, and sketch the quadrilateral ABDC.

[3]



(b) Find $\arg(\omega)$.

[2]

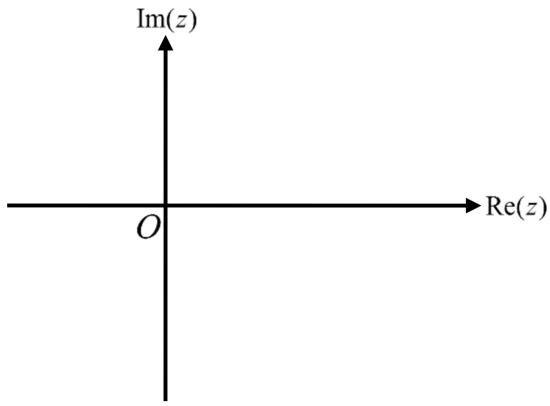
(c) Find the area of the quadrilateral ABDC.

[2]

4. In an Argand diagram with the origin O, the points A, B and C represent the complex numbers $z=-3+6\text{i}$, $z-6-15\text{i}$ and $(18+9\text{i})^*$ respectively.

(a) Sketch the points A, B and C on the following Argand diagram, and sketch the triangle ABC.

[3]



(b) Find $\arg(z-6-15\text{i})$.

[2]

(c) Find the exact area of the triangle ABC.

[2]

22 Paper 1 – Forms of Complex Numbers

Example

Consider the complex numbers $z_1 = \text{cis}\frac{\pi}{6}$ and $z_2 = 6\text{cis}\frac{2\pi}{3}$.

(a) Express $z_1 z_2$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the imaginary part of $z_1 z_2$.

[2]

Solution

(a) (i) $z_1 z_2 = \left(\text{cis}\frac{\pi}{6}\right)\left(6\text{cis}\frac{2\pi}{3}\right)$

$z_1 z_2 = (1)(6)\text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$ (M1) for valid approach

$z_1 z_2 = 6\text{cis}\frac{5\pi}{6}$ A1

(ii) $z_1 z_2 = 6e^{\frac{5\pi}{6}i}$ A1

[3]

(b) The imaginary part of $z_1 z_2$

$= 6\sin\frac{5\pi}{6}$ (M1) for valid approach

$= 3$ A1

[2]

6

Exercise 22

1. Consider the complex numbers $z_1 = 12\text{cis}\frac{7\pi}{6}$ and $z_2 = 4\text{cis}\frac{\pi}{2}$.

(a) Express $\frac{z_1}{z_2}$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the real part of $\frac{z_1}{z_2}$.

[2]

2. Consider the complex numbers $z_1 = 18\sqrt{3}\text{cis}\left(-\frac{\pi}{6}\right)$ and $z_2 = \frac{1}{9}\text{cis}\frac{\pi}{3}$.

(a) Express $z_1 z_2$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

(b) Hence, find the real part of $z_1 z_2$.

[2]

3. Consider the complex numbers $z_1 = 2\text{cis}\frac{\pi}{12}$ and $z_2 = 3\text{cis}\frac{\pi}{4}$.

(a) (i) Express z_1^2 in the form $r\text{cis}\theta$.

(ii) Hence, find the imaginary part of z_1^2.

[4]

(b) Express $z_1^2 z_2$ in the form

(i) $r\text{cis}\theta$;

(ii) $re^{i\theta}$.

[3]

4. Consider the complex numbers $z_1 = \frac{1}{3}\text{cis}\frac{\pi}{6}$ and $z_2 = \frac{1}{9}\text{cis}\frac{11\pi}{12}$.

(a) (i) Express $z_1^{\,4}$ in the form $r\text{cis}\theta$.

(ii) Hence, find the real part of $z_1^{\,4}$.

[4]

(b) Express $\frac{z_2}{z_1^{\,4}}$ in the form

(i) $r\text{cis}\theta$;

(ii) $r\text{e}^{\text{i}\theta}$.

[3]

23 Paper 1 – Complex Roots of Quadratic Equations

Example

A quadratic function is given by $f(x)=x^2-6x+58$. It is given that the range of $f(x)$ is $\{y:y\geq 49\}$.

(a) Explain why there is no real root for the equation $f(x)=0$.

[1]

(b) Find the complex roots of the equation $f(x)=0$, giving the answer in the form $a+b\text{i}$, where a, $b\in\mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

Solution

(a) The range of $f(x)$ is $\{y:y\geq 49\}$, means the graph of $f(x)$ does not have any x-intercept. R1

[1]

(b) $x^2-6x+58=0$

$x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(58)}}{2(1)}$ (A1) for substitution

$x=\dfrac{6\pm\sqrt{-196}}{2}$ (A1) for simplification

$x=\dfrac{6\pm\sqrt{196}\text{i}}{2}$

$x=3\pm 7\text{i}$ A1

[3]

(c) 14 A1

[1]

Exercise 23

1. A quadratic function is given by $f(x) = -x^2 + 4x - 29$. It is given that the range of $f(x)$ is $\{y : y \le -25\}$.

(a) Explain why there is no real root for the equation $f(x) = 0$.

[1]

(b) Find the complex roots of the equation $f(x) = 0$, giving the answer in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

2. A quadratic function is given by $f(x) = (x+5)^2 + 64$.

(a) Explain why there is no real root for the equation $f(x) = 0$.

[1]

(b) Find the complex roots of the equation $f(x) = 0$, giving the answer in the form $a + b\mathrm{i}$, where a, $b \in \mathbb{Q}$.

[3]

(c) If the above two complex roots are located on an Argand diagram, write down the distance between the roots.

[1]

3. A quadratic function is given by $f(x) = ax^2 + bx + c$. It is given that the complex roots of $f(x) = 0$ are $4 + 13\mathrm{i}$ and $4 - 13\mathrm{i}$.

(a) Write down the values of

(i) $(4+13\mathrm{i}) + (4-13\mathrm{i})$;

(ii) $(4+13\mathrm{i})(4-13\mathrm{i})$.

[2]

(b) Hence, find the expression of $f(x)$, giving the answer in terms of a.

[3]

The graph of $f(x)$ passes through $(4, 169)$.

(c) Find the value of a.

[2]

4. A quadratic function is given by $f(x)=ax^2+bx+c$. It is given that the complex roots of $f(x)=0$ are $-\frac{1}{2}+2\text{i}$ and $-\frac{1}{2}-2\text{i}$.

(a) Write down the values of

(i) $\left(-\frac{1}{2}+2\text{i}\right)+\left(-\frac{1}{2}-2\text{i}\right)$;

(ii) $\left(-\frac{1}{2}+2\text{i}\right)\left(-\frac{1}{2}-2\text{i}\right)$.

[2]

(b) Hence, find the expression of $f(x)$, giving the answer in terms of a.

[3]

The graph of $f(x)$ passes through $(0,-17)$.

(c) Find the value of a.

[2]

24 Paper 2 – Applications of Complex Numbers

Example

Two alternating current electrical sources are given as $V_1 = 4\cos(4t + 0.1)$ and $V_2 = 3\cos 4t$ respectively, where t represents time in seconds. The total voltage V is given by $V = V_1 + V_2$.

(a) Write down the amplitude of

(i) V_1;

(ii) V_2.

[2]

(b) Find the period of V_2.

[2]

It is given that $V_1 + V_2 = \text{Re}(e^{4ti}(z + w))$, z, $w \in \mathbb{C}$.

(c) Find the expression of $z + w$.

[3]

(d) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z + w = Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $V_1 + V_2 = \text{Re}(e^{4ti}(z + w))$, express V in the form $A\cos(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(g) Hence, find the total voltage when $t = 1$.

[2]

Solution

(a) (i) 4 A1

(ii) 3 A1

[2]

(b) The period of V_2

$= \frac{2\pi}{4}$ (M1) for valid approach

$= \frac{\pi}{2}$ s A1

[2]

(c) $V_1 + V_2 = 4\cos(4t + 0.1) + 3\cos 4t$

$V_1 + V_2 = \text{Re}(4e^{(4t+0.1)\text{i}}) + \text{Re}(3e^{4t\text{i}})$ (M1) for valid approach

$V_1 + V_2 = \text{Re}(4e^{(4t+0.1)\text{i}} + 3e^{4t\text{i}})$ (A1) for correct approach

$V_1 + V_2 = \text{Re}(e^{4t\text{i}}(4e^{0.1\text{i}} + 3))$

$\therefore z + w = 4e^{0.1\text{i}} + 3$ A1

[3]

(d) (i) $z = 4e^{0.1\text{i}}$

$z = 4(\cos 0.1 + \text{i}\sin 0.1)$ A1

(ii) $w = 3$

$w = 3(\cos 0 + \text{i}\sin 0)$ A1

[2]

(e) (i) $z + w = 4(\cos 0.1 + \text{i}\sin 0.1) + 3(\cos 0 + \text{i}\sin 0)$

$z + w = (4\cos 0.1 + 3\cos 0)$

$+\text{i}(4\sin 0.1 + 3\sin 0)$ (M1) for valid approach

$z + w = 6.980016661 + 0.3993336666\text{i}$ (A1) for correct values

$L = \sqrt{6.980016661^2 + 0.3993336666^2}$ M1

$L = 6.991430466$

$L = 6.99$ A1

(ii) $\alpha = \arctan\frac{0.3993336666}{6.980016661}$ M1

$\alpha = 0.0571486937$

$\alpha = 0.0571$ A1

[6]

(f) $V_1 + V_2 = \text{Re}(e^{4ti}(z+w))$

$V_1 + V_2 = \text{Re}(e^{4ti} \cdot 6.991430466e^{0.0571486937i})$ (M1) for substitution

$V_1 + V_2 = \text{Re}(6.991430466e^{4ti+0.0571486937i})$ (A1) for correct approach

$V_1 + V_2 = 6.991430466\cos(4t + 0.0571486937)$

$V_1 + V_2 = 6.99\cos(4t + 0.0571)$ A1

[3]

(g) The required total voltage

$= 6.991430466\cos(4(1) + 0.0571486937)$ (M1) for substitution

$= -4.260226648$ V

$= -4.26$ V A1

[2]

6

Exercise 24

1. Two sound waves are given as $S_1 = 2\sin(6t - 0.1)$ and $S_2 = 3\sin(6t + 0.25)$ respectively, where S_1 and S_2 represent the amplitudes of the two sound waves respectively, in millimetres, and t represents time in seconds. The total amplitude S is given by $S = S_1 + S_2$.

(a) Write down the amplitude of

(i) S_1;

(ii) S_2.

[2]

(b) Find the period of S_2.

[2]

It is given that $S_1 + S_2 = \text{Im}(e^{6ti}(z+w))$, z, $w \in \mathbb{C}$.

(c) Find the expression of $z + w$.

[3]

(d) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z+w=Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $S_1+S_2=\text{Im}(e^{6\pi i}(z+w))$, express S in the form $A\sin(Bt+C)$, A, B, $C\in\mathbb{R}$.

[3]

(g) Hence, write down the minimum total amplitude.

[1]

2. Two waves are given as $W_1=5\cos(\pi t-0.9)$ and $W_2=7\cos(\pi t-1.3)$ respectively, where W_1 and W_2 represent the amplitudes of the two waves respectively. t represents time in seconds. The total amplitude W is given by $W=W_1+W_2$.

(a) Write down the amplitude of

(i) W_1;

(ii) W_2.

[2]

(b) Find the period of W_2.

[2]

It is given that $W_1+W_2=\text{Re}(e^{\pi ti}(z+w))$, z, $w\in\mathbb{C}$.

(c) Find the expression of $z+w$.

[3]

(d) Express the following in the form $r(\cos\theta+i\sin\theta)$:

(i) z

(ii) w

[2]

(e) It is given that $z+w=Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(f) Using $W_1+W_2=\text{Re}(e^{\pi ti}(z+w))$, express W in the form $A\cos(Bt+C)$, A, B, $C\in\mathbb{R}$.

[3]

(g) Hence, find t when $W=0$, $1<t<2$.

[2]

3. Two sound waves are given as $S_1 = 8\cos(10t + 0.05)$ and S_2 respectively, where S_1 and S_2 represent the amplitudes of the two sound waves respectively, in millimetres, and t represents time in seconds. The total amplitude S is given by
$S = S_1 + S_2 = 10\cos(10t + 0.15)$.

(a) For S, write down its

(i) amplitude;

(ii) period.

[2]

It is given that $S_2 = \text{Re}(e^{10ti}(z - w))$, z, $w \in \mathbb{C}$.

(b) Find the expression of $z - w$.

[3]

(c) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(d) It is given that $z - w = Le^{i\alpha}$. Find

(i) L;

(ii) α.

[6]

(e) Express S_2 in the form $A\cos(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(f) Hence, find the value of t when $S_2 = 1.5$, $9.5 < t < 10$.

[2]

4. Two alternating current electrical sources are given as $V_1 = 7\sin(2\pi t - 0.95)$ and V_2 respectively, where t represents time in seconds. The total voltage V is given by $V = V_1 + V_2 = 6.3\sin(2\pi t - 0.5)$.

(a) For V_1, write down its

(i) amplitude;

(ii) period.

[2]

It is given that $V_2 = \text{Im}(e^{2\pi t\text{i}}(z - w))$, z, $w \in \mathbb{C}$.

(b) Find the expression of $z - w$.

[3]

(c) Express the following in the form $r(\cos\theta + i\sin\theta)$:

(i) z

(ii) w

[2]

(d) It is given that $z - w = Le^{\text{i}\alpha}$. Find

(i) L;

(ii) α.

[6]

(e) Express V_2 in the form $A\sin(Bt + C)$, A, B, $C \in \mathbb{R}$.

[3]

(f) Hence, find the range of values of t when $V_2 > 2$, $0.5 \le t \le 1.5$.

[3]

Chapter

7

Matrices

SUMMARY POINTs

✓ Terminologies of matrices:

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$: A $m\times n$ matrix with m rows and n columns

a_{ij} : Element on the i th row and the j th column

$$\mathbf{I}=\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$: Identity matrix

$$\mathbf{0}=\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$: Zero matrix

SUMMARY POINTs

✓ Terminologies of matrices:

$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$: Diagonal matrix

$|\mathbf{A}| = \det(\mathbf{A})$: Determinant of $\mathbf{A}$

$\mathbf{A}$ is non-singular if $\det(\mathbf{A}) \neq 0$

$\mathbf{A}^{-1}$: Inverse of $\mathbf{A}$

$\mathbf{A}^{-1}$ exists if $\mathbf{A}$ is non-singular

✓ For any 2×2 square matrices $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

1. $|\mathbf{A}| = \det(\mathbf{A}) = ad - bc$: Determinant of $\mathbf{A}$
2. $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$: Inverse of $\mathbf{A}$

✓ Operations of matrices:

1. $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}$
2. $k\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ka_{11} & \cdots & ka_{1n} \\ \vdots & \ddots & \vdots \\ ka_{m1} & \cdots & ka_{mn} \end{pmatrix}$
3. $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$: The element on the i th row and the j th column of $\mathbf{C} = \mathbf{AB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}\begin{pmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{pmatrix}$, where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $m \times n$, $n \times k$ and $m \times k$ matrices respectively

SUMMARY POINTs

✓ A 2×2 system $\begin{cases} ax+by=c \\ dx+ey=f \end{cases}$ can be expressed as $\mathbf{AX}=\mathbf{B}$, where $\mathbf{X}=\begin{pmatrix} x \\ y \end{pmatrix}$ can be solved by $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}=\begin{pmatrix} a & b \\ d & e \end{pmatrix}^{-1}\begin{pmatrix} c \\ f \end{pmatrix}$

✓ A 3×3 system $\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kz=l \end{cases}$ can be expressed as $\mathbf{AX}=\mathbf{B}$, where $\mathbf{X}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be solved by $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}=\begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}^{-1}\begin{pmatrix} d \\ h \\ l \end{pmatrix}$

✓ Eigenvalues and eigenvectors of $\mathbf{A}$:

1. $\det(\mathbf{A}-\lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{A}$
2. Solution(s) of $\det(\mathbf{A}-\lambda\mathbf{I})=0$: Eigenvalue(s) of $\mathbf{A}$
3. $\mathbf{v}$: Eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, which satisfies $\mathbf{Av}=\lambda\mathbf{v}$

✓ Diagonalization of $\mathbf{A}$:

1. $\mathbf{D}=\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$: Diagonal matrix of the eigenvalues of $\mathbf{A}$
2. $\mathbf{V}=\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{pmatrix}$: A matrix of the eigenvectors of $\mathbf{A}$
3. $\mathbf{A}=\mathbf{VDV}^{-1}\Rightarrow\mathbf{A}^n=\mathbf{VD}^n\mathbf{V}^{-1}$

SUMMARY POINTs

✓ Two-dimensional transformation matrices:

1. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$: Reflection about the x-axis
2. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$: Reflection about the y-axis
3. $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$: Reflection about the line $y = mx$, where $m = \tan\theta$
4. $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$: Vertical stretch with scale factor k
5. $\begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}$: Horizontal stretch with scale factor k
6. $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$: Enlargement about the origin with scale factor k
7. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ anticlockwise about the origin
8. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$: Rotation with positive angle θ clockwise about the origin
9. Area of the image $= |\det(T)| \cdot$ Area of the object, where T is the transformation matrix

Solutions of Chapter 7

25 Paper 1 – Matrix Algebra

Example

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & 1 \\ -2 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3x^2 & 4x^2 \\ 2x^3 & 3x^3 \end{pmatrix}$.

(a) Write down $\det \mathbf{A}$.

[1]

(b) Find $\det \mathbf{B}$.

[2]

(c) Solve the equation $4\det \mathbf{A} = \det \mathbf{B}$.

[2]

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Solution

(a) 8 — A1

[1]

(b) $\det \mathbf{B} = (3x^2)(3x^3) - (4x^2)(2x^3)$ — (A1) for substitution

$\det \mathbf{B} = 9x^5 - 8x^5$

$\det \mathbf{B} = x^5$ — A1

[2]

(c) $4\det \mathbf{A} = \det \mathbf{B}$

$\therefore 4(8) = x^5$ — (M1) for setting equation

$x^5 = 32$

$x = 2$ — A1

[2]

Exercise 25

1. Let $\mathbf{A}=\begin{pmatrix}2 & 4 & 1\\ -3 & -5 & 3\\ 1 & 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}2x^2 & 4\\ x & x\end{pmatrix}$.

(a) Write down $\det\mathbf{A}$. [1]

(b) Find $\det\mathbf{B}$. [2]

(c) Solve the equation $(11-\det\mathbf{A})x=\det\mathbf{B}$. [2]

2. Let $\mathbf{A}=\begin{pmatrix}3 & 1 & -1\\ 0 & 2 & -5\\ 1 & 0 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1+x & x\\ -2x & 1-x\end{pmatrix}$.

(a) Write down $\det\mathbf{A}$. [1]

(b) Find $\det\mathbf{B}$. [2]

(c) Solve the equation $\det\mathbf{A}+\det\mathbf{B}-5x=0$. [3]

3. Let $\mathbf{A}=\begin{pmatrix}4 & 1\\ 3e^x & e^{2x}\end{pmatrix}$.

(a) Find $\det\mathbf{A}$. [2]

(b) Solve the equation $\det\mathbf{A}-1=0$. [3]

4. Let $\mathbf{A}=\begin{pmatrix}\ln x & 3\\ -2 & \ln x\end{pmatrix}$.

(a) Find $\det\mathbf{A}$. [2]

(b) Solve the equation $\det\mathbf{A}=5\ln x$, giving the answer(s) in terms of e. [4]

26 Paper 1 – Inverse of a Matrix

Example

Let $\mathbf{A}=\begin{pmatrix}3 & 4 & 5\\ 2 & -3 & -4\\ 2 & -1 & 0\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB}+\mathbf{I}=\begin{pmatrix}2 & 3 & 1\\ 0 & 1 & -1\\ 3 & -3 & 2\end{pmatrix}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

Solution

(a) $\mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{6} & \frac{5}{24} & \frac{1}{24}\\ \frac{1}{3} & \frac{5}{12} & -\frac{11}{12}\\ -\frac{1}{6} & -\frac{11}{24} & \frac{17}{24}\end{pmatrix}$ A2

[2]

(b) $\mathbf{AB}+\mathbf{I}=\begin{pmatrix}2 & 3 & 1\\ 0 & 1 & -1\\ 3 & -3 & 2\end{pmatrix}$

$\mathbf{AB}=\begin{pmatrix}1 & 3 & 1\\ 0 & 0 & -1\\ 3 & -3 & 1\end{pmatrix}$

$\mathbf{B}=\mathbf{A}^{-1}\begin{pmatrix}1 & 3 & 1\\ 0 & 0 & -1\\ 3 & -3 & 1\end{pmatrix}$ (M1) for valid approach

$$\mathbf{B}=\begin{pmatrix} \frac{7}{24} & \frac{3}{8} & 0 \\ -\frac{29}{12} & \frac{15}{4} & -1 \\ \frac{47}{24} & -\frac{21}{8} & 1 \end{pmatrix}$$ A2

[3]

Exercise 26

1. Let $\mathbf{A}=\begin{pmatrix} -6 & -3 & 1 \\ 1 & 4 & -2 \\ 1 & -2 & 1 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB}+\begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 3 \\ -1 & 0 & -3 \end{pmatrix}=2\mathbf{I}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

2. Let $\mathbf{A}=\begin{pmatrix} 3 & 1 & 0 \\ 2 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{AB}=\begin{pmatrix} -8 & 5 & 3 \\ 2 & 6 & 7 \\ 5 & -4 & -4 \end{pmatrix}-5\mathbf{I}$, where $\mathbf{B}$ and $\mathbf{I}$ are 3×3 matrices.

(b) Find $\mathbf{B}$.

[3]

3. Let $\mathbf{A} = \begin{pmatrix} 2 & 3 & 2 \\ -3 & 3 & -4 \\ 2 & 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 10 & -7 & 2 \\ 5 & -9 & 6 \\ -4 & 3 & 8 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{A}^{-1}\mathbf{C}\mathbf{A} = \frac{1}{2}\mathbf{B}$, where $\mathbf{C}$ is a 3×3 matrix.

(b) Find $\mathbf{C}$.

[3]

4. Let $\mathbf{A} = \begin{pmatrix} 0 & 4 & 5 \\ 0 & 2 & 3 \\ 1 & -5 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & -3 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

It is given that $\mathbf{A}\mathbf{C}\mathbf{A}^{-1} = \mathbf{B}^3$, where $\mathbf{C}$ is a 3×3 matrix.

(b) Find $\mathbf{C}$.

[3]

7

27 Paper 1 – Systems of Equations

Example

Let $\mathbf{A}=\begin{pmatrix}1 & 2 & 3\\ 0 & -5 & -6\\ 0 & -3 & -2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}2\\ 7\\ 11\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$.

[2]

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$.

[3]

Solution

(a) $\mathbf{A}^{-1}=\begin{pmatrix}1 & \frac{5}{8} & -\frac{3}{8}\\ 0 & \frac{1}{4} & -\frac{3}{4}\\ 0 & -\frac{3}{8} & \frac{5}{8}\end{pmatrix}$ A2

[2]

(b) $\mathbf{AX}=\mathbf{B}$

$\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ (M1) for valid approach

$\mathbf{X}=\begin{pmatrix}\frac{9}{4}\\ -\frac{13}{2}\\ \frac{17}{4}\end{pmatrix}$ A2

[3]

Exercise 27

1. Let $\mathbf{A}=\begin{pmatrix}2&1&6\\4&5&1\\1&2&4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-3\\0\\4\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$. [2]

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$. [3]

2. Let $\mathbf{A}=\begin{pmatrix}2&0&-5\\1&2&-8\\6&-1&-3\end{pmatrix}$.

(a) Write down $\mathbf{A}^{-1}$. [2]

(b) Hence, solve the system $\begin{cases}2x-5z=740\\x+2y-8z=592\\6x-y-3z=-444\end{cases}$. [4]

3. Let $\mathbf{A}=\begin{pmatrix}3&2&1\\2&a&1\\1&1&0\end{pmatrix}$, where $a\in\mathbb{Z}$. It is given that $\mathbf{A}^{-1}=\begin{pmatrix}1&-1&0\\-1&1&1\\0&1&-2\end{pmatrix}$.

(a) Find a. [2]

It is also given that $\mathbf{B}=\begin{pmatrix}0\\6\\9\end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$. [3]

4. Let $\mathbf{A}=\begin{pmatrix} p & 16 & 16 \\ 8 & -8 & q \\ 8 & -16 & -16 \end{pmatrix}$, where p, $q \in \mathbb{Z}$. It is given that $\mathbf{A}^{-1}=\begin{pmatrix} \frac{1}{16} & 0 & \frac{1}{16} \\ \frac{1}{64} & \frac{1}{16} & -\frac{5}{64} \\ \frac{1}{64} & -\frac{1}{16} & \frac{3}{64} \end{pmatrix}$.

(a) Find p and q.

[3]

It is also given that $\mathbf{B}=\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{X}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) Solve $\mathbf{X}$ in the equation $\mathbf{AX}=\mathbf{B}$.

[3]

28 Paper 1 – Transformation Matrices

Example

The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point $(2, 4)$ to the point P. Find the coordinates of P.

[2]

Solution

(a) $\mathbf{ST} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

$\mathbf{ST} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$ A2

[2]

(b) Rotation anticlockwise of $\frac{2\pi}{3}$ radians about the origin. A1

[1]

(c) $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ (M1) for valid approach

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4.464101615 \\ -0.2679491924 \end{pmatrix}$

Thus, the coordinates of P are $(-4.46, -0.268)$. A1

[2]

Exercise 28

1. The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point $(3, -5)$ to the point P. Find the coordinates of P.

[2]

2. The transformation $\mathbf{S}$ and $\mathbf{T}$ are represented by the matrices $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ respectively.

(a) Find $\mathbf{ST}$.

[2]

(b) Describe the transformation represented by $\mathbf{ST}$.

[1]

(c) $\mathbf{ST}$ transforms the point P to the point $(2, 1)$. Find the coordinates of P.

[3]

3. Let $\mathbf{T} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.

(a) Describe the transformation represented by $\mathbf{T}$.

[1]

(b) $\mathbf{T}$ transforms the point $(4, 4)$ to the point P. Find the coordinates of P.

[2]

(c) Write down the smallest positive integer n such that $\mathbf{T}^n = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix.

[2]

4. Let $\mathbf{T} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$.

(a) Describe the transformation represented by $\mathbf{T}$.

[1]

(b) $\mathbf{T}$ transforms the point P to the point $(-2, 2\sqrt{3})$. Find the coordinates of P.

[3]

(c) Write down the smallest positive integer n such that $\mathbf{T}^n = \mathbf{I}$, where $\mathbf{I}$ is a 2×2 identity matrix.

[2]

7

29 Paper 2 – Eigenvalues and Eigenvectors

Example

The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}-2 & 1\\ -5 & 4\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1 < \lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\det(\mathbf{A})=\alpha\lambda_1\lambda_2$, where $\alpha \in \mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

Solution

(a) The characteristic polynomial of $\mathbf{A}$

$=\det(\mathbf{A}-\lambda\mathbf{I})$

$=\begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix}$ (M1) for valid approach

$=(-2-\lambda)(4-\lambda)-(1)(-5)$

$=-8+2\lambda-4\lambda+\lambda^2+5$

$=\lambda^2-2\lambda-3$ A1

[2]

(b) $\lambda_1=-1$, $\lambda_2=3$ A2

[2]

(c) $\mathbf{v}_1=\begin{pmatrix}1\\1\end{pmatrix}$, $\mathbf{v}_2=\begin{pmatrix}1\\5\end{pmatrix}$ A2

[2]

(d) $\det(\mathbf{A})=\alpha\lambda_1\lambda_2$

$\therefore -3=\alpha(-1)(3)$ (M1) for setting equation

$\alpha=1$ A1

[2]

(e) (i) $\begin{pmatrix}1 & 1\\1 & 5\end{pmatrix}$ A1

(ii) $\begin{pmatrix}(-1)^n & 0\\0 & 3^n\end{pmatrix}$ A2

[3]

(f) $\mathbf{A}^n=\begin{pmatrix}1 & 1\\1 & 5\end{pmatrix}\begin{pmatrix}(-1)^n & 0\\0 & 3^n\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 5\end{pmatrix}^{-1}$

$\mathbf{A}^n=\begin{pmatrix}(-1)^n & 3^n\\(-1)^n & 5\cdot3^n\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 5\end{pmatrix}^{-1}$ A1

$\mathbf{A}^n=\begin{pmatrix}(-1)^n & 3^n\\(-1)^n & 5\cdot3^n\end{pmatrix}\begin{pmatrix}\frac{5}{4} & -\frac{1}{4}\\-\frac{1}{4} & \frac{1}{4}\end{pmatrix}$ (A1) for correct approach

$\mathbf{A}^n=\begin{pmatrix}\frac{5}{4}(-1)^n-\frac{1}{4}\cdot3^n & -\frac{1}{4}(-1)^n+\frac{1}{4}\cdot3^n\\\frac{5}{4}(-1)^n-\frac{5}{4}\cdot3^n & -\frac{1}{4}(-1)^n+\frac{5}{4}\cdot3^n\end{pmatrix}$ A1

[3]

Exercise 29

1. The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}-2 & -3\\ 1 & 2\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{A}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\det(\mathbf{A})=\dfrac{\alpha}{\lambda_1\lambda_2}$, where $\alpha\in\mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n=\mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{A}^n$ in terms of n.

[3]

2. The matrix $\mathbf{A}$ is defined by $\mathbf{A}=\begin{pmatrix}9 & -4\\ 2 & 3\end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{A}$, where $\lambda_1<\lambda_2$.

(a) Find $\det(\mathbf{A}-\lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{A}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $3\det(\mathbf{A}) + \alpha\lambda_1\lambda_2 = 0$, where $\alpha \in \mathbb{R}$.

(d) Find α.

[2]

It is given that $\mathbf{A}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, find $\mathbf{A}^{10}$, giving the entries in exact values.

[3]

3. The matrix $\mathbf{M}$ is defined by $\mathbf{M} = \begin{pmatrix} -1 & \frac{1}{16} \\ -35 & 2 \end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{M}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{M}^n$ in terms of n.

[3]

Let $f(n)$ be the first diagonal entry of $\mathbf{M}^n$.

(f) Write down $\lim_{n\to\infty} f(n)$.

[1]

4. The matrix $\mathbf{M}$ is defined by $\mathbf{M} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find the characteristic polynomial of $\mathbf{M}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{M}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{M}^n$ in terms of n.

[3]

Let $g(n)$ be the last diagonal entry of $\mathbf{M}^n$.

(f) Write down $\lim_{n\to\infty} g(n)$.

[1]

30 Paper 2 – Miscellaneous Problems

Example

The function f is defined by $f(x)=ax^2+bx+c$, where a, b, $c\in\mathbb{Z}$. It is given that the graph of f passes through $(-10,540)$, $(10,500)$ and $(20,1980)$.

(a) (i) Show that $100a-10b+c=540$.

(ii) Write down the other two equations in a, b and c.

[3]

The above three equations can be expressed in a matrix equation $\mathbf{AX}=\mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Find

(i) the equation of the axis of symmetry;

(ii) the y-coordinate of the vertex.

[4]

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Solution

(a)	(i)	$540 = a(-10)^2 + b(-10) + c$	A1
		$100a - 10b + c = 540$	AG
	(ii)	$100a + 10b + c = 500$	A1
		$400a + 20b + c = 1980$	A1
			[3]
(b)	(i)	$\mathbf{A} = \begin{pmatrix} 100 & -10 & 1 \\ 100 & 10 & 1 \\ 400 & 20 & 1 \end{pmatrix}$	A1
	(ii)	$\mathbf{B} = \begin{pmatrix} 540 \\ 500 \\ 1980 \end{pmatrix}$	A1
	(iii)	$\mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{600} & -\frac{1}{200} & \frac{1}{300} \\ -\frac{1}{20} & \frac{1}{20} & 0 \\ \frac{1}{3} & 1 & -\frac{1}{3} \end{pmatrix}$	A2
			[4]
(c)	$a = 5$, $b = -2$ and $c = 20$		
	For any one correct answer		A1
	For all correct answers		A1
			[2]
(d)	(i)	The equation of the axis of symmetry:	
		$x = -\frac{-2}{2(5)}$	(A1) for substitution
		$x = \frac{1}{5}$	A1
	(ii)	The y-coordinate of the vertex	
		$= 5\left(\frac{1}{5}\right)^2 - 2\left(\frac{1}{5}\right) + 20$	(M1) for substitution
		$= \frac{99}{5}$	A1
			[4]

Exercise 30

1. The function f is defined by $f(x)=ax^2+bx+c$, where a, b, $c\in\mathbb{Z}$. It is given that the graph of f passes through $(50, 3600)$, $(20, -900)$ and $(5, -1125)$.

(a) (i) Show that $2500a+50b+c=3600$.

(ii) Write down the other two equations in a, b and c.

[3]

The above three equations can be expressed in a matrix equation $\mathbf{AX}=\mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Find

(i) the x-intercept(s) of the graph of f;

(ii) the y-coordinate of the vertex.

[5]

2. The function f is defined by $f(x)=ax^3+bx^2+cx+d$, where a, b, c, $d\in\mathbb{Z}$. It is given that the graph of f passes through $(0,-384)$, $(2,-840)$, $(6,-2520)$ and $(10,-5544)$.

(a) (i) Show that $d=-384$.

(ii) Show that $4a+2b+c=-228$.

(iii) Write down the other two equations in a, b and c.

[4]

The above three equations can be expressed in a matrix equation $\mathbf{AX}=\mathbf{B}$, where $\mathbf{A}$ is a 3×3 matrix, and $\mathbf{X}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ and $\mathbf{B}$ are two 3×1 matrices.

(b) Write down

(i) $\mathbf{A}$;

(ii) $\mathbf{B}$;

(iii) $\mathbf{A}^{-1}$.

[4]

(c) Hence, find the values of a, b and c.

[2]

(d) Write down

(i) the x-intercept(s) of the graph of f;

(ii) the y-intercept of the graph of f.

[4]

3. Let $\mathbf{M}=\begin{pmatrix}1&0\\5&1\end{pmatrix}$.

(a) (i) Find $\mathbf{M}^2$.

(ii) Find $\mathbf{M}^3$.

(iii) By using the above results, write down $\mathbf{M}^{50}$.

[5]

Let $s(n)=\mathbf{M}+\mathbf{M}^2+\mathbf{M}^3+\cdots+\mathbf{M}^n$, where $n\geq1$.

(b) (i) Write down $s(2)$.

(ii) Write down $s(3)$.

(iii) By using the above results, find $s(50)$.

[6]

Let $p(n)=\mathbf{M}\times\mathbf{M}^2\times\mathbf{M}^3\times\cdots\times\mathbf{M}^n$, where $n\geq1$.

(c) Find $p(50)$.

[4]

4. Let $\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$.

(a) (i) Find $\mathbf{A}^2$.

(ii) Find $\mathbf{A}^3$.

(iii) By using the above results, write down $\mathbf{A}^{30}$.

[5]

Let $\mathbf{B} = \mathbf{A} + \begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}$.

(b) (i) Show that $\mathbf{B}^2 = \begin{pmatrix} 1 & 21 \\ 0 & 4 \end{pmatrix}$.

(ii) Find $\mathbf{B}^3$.

It is given that $\mathbf{B}^4$ can be expressed as $\begin{pmatrix} 1 & 7+14+28+56 \\ 0 & 16 \end{pmatrix}$.

(iii) Find $\mathbf{B}^{30}$.

[8]

(c) Explain why $\det(\mathbf{B}^n) = \det(\mathbf{A}^n) + \det\left(\begin{pmatrix} 0 & 3 \\ 0 & 1 \end{pmatrix}^n\right)$ is not always true for $n \geq 1$, $n \in \mathbb{Z}$.

[1]

7

Chapter

8

Vectors

SUMMARY POINTs

✓ Terminologies of vectors:

$\overrightarrow{\mathrm{AB}}$: Vector of length AB with initial point A and terminal point B

$\overrightarrow{\mathrm{OP}}$: Position vector of P, where O is the origin

$\left|\overrightarrow{\mathrm{AB}}\right|$: Magnitude (length) of $\overrightarrow{\mathrm{AB}}$

$\hat{\mathbf{v}} = \frac{1}{|\mathbf{v}|}\mathbf{v}$: Unit vector parallel to $\mathbf{v}$, with $|\hat{\mathbf{v}}| = 1$

$\mathbf{0}$: Zero vector

$\mathbf{i}$: Unit vector along the positive x-axis

$\mathbf{j}$: Unit vector along the positive y-axis

$\mathbf{k}$: Unit vector along the positive z-axis

✓ A vector $\mathbf{v}$ can be expressed as $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ or $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

SUMMARY POINTs

✓ Properties of vectors:

1. $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
2. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \pm \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \pm v_1 \\ u_2 \pm v_2 \\ u_3 \pm v_3 \end{pmatrix}$
3. $\mathbf{v}$ and $k\mathbf{v}$ are in the same direction if $k > 0$
4. $\mathbf{v}$ and $k\mathbf{v}$ are in opposite direction if $k < 0$
5. $k\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$

✓ Properties of the scalar product $\mathbf{u}\cdot\mathbf{v}$ of $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = |\mathbf{u}||\mathbf{v}|\cos\theta$
2. $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$
3. $\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$
4. $\mathbf{u}$ and $\mathbf{v}$ are in the same direction if $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|$
5. $\mathbf{u}$ and $\mathbf{v}$ are in opposite direction if $\mathbf{u}\cdot\mathbf{v} = -|\mathbf{u}||\mathbf{v}|$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $\mathbf{u}\cdot\mathbf{v} = 0$
7. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
8. $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2$

SUMMARY POINTs

✓ Properties of the vector product $\mathbf{u}\times\mathbf{v}$ of $\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$:

1. $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{pmatrix}=|\mathbf{u}||\mathbf{v}|\sin\theta\hat{\mathbf{n}}$, where $\hat{\mathbf{n}}//(\mathbf{u}\times\mathbf{v})$
2. $\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=\mathbf{0}$
3. $\mathbf{i}\times\mathbf{j}=\mathbf{k}$, $\mathbf{j}\times\mathbf{k}=\mathbf{i}$ and $\mathbf{k}\times\mathbf{i}=\mathbf{j}$
4. $\mathbf{j}\times\mathbf{i}=-\mathbf{k}$, $\mathbf{k}\times\mathbf{j}=-\mathbf{i}$ and $\mathbf{i}\times\mathbf{k}=-\mathbf{j}$
5. $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u}\times\mathbf{v}=0$
6. $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if $|\mathbf{u}\times\mathbf{v}|=|\mathbf{u}||\mathbf{v}|$
7. $\mathbf{u}\times\mathbf{v}=-(\mathbf{v}\times\mathbf{u})$

✓ The area of the parallelogram with adjacent sides $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$ is $\left|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AD}}\right|$

✓ The area of the triangle with adjacent sides $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AD}}$ is $\frac{1}{2}\left|\overrightarrow{\mathrm{AB}}\times\overrightarrow{\mathrm{AD}}\right|$

✓ Forms of the straight line with fixed point $\mathrm{A}(a_1, a_2, a_3)$ and direction vector $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$:

1. $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+t\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, $t\in\mathbb{R}$
2. $\begin{cases}x=a_1+b_1t\\y=a_2+b_2t\\z=a_3+b_3t\end{cases}$: Parametric form

✓ Vector components:

1. $\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ parallel to $\mathbf{v}$
2. $\frac{|\mathbf{u}\times\mathbf{v}|}{|\mathbf{v}|}$: Vector component of $\mathbf{u}$ perpendicular to $\mathbf{v}$

Solutions of Chapter 8

31 Paper 1 – Intersection Between Lines

Example

Consider the lines L_1 and L_2 defined by $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + s \begin{pmatrix} k \\ -3 \\ k-2 \end{pmatrix}$ and

$L_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -9 \end{pmatrix} + t \begin{pmatrix} k-2 \\ 2 \\ -1 \end{pmatrix}$, $k \in \mathbb{R}$. It is given that L_1 and L_2 are perpendicular. Find the possible values of k.

[4]

Solution

$\because L_1$ and L_2 are perpendicular.

$\therefore \begin{pmatrix} k \\ -3 \\ k-2 \end{pmatrix} \cdot \begin{pmatrix} k-2 \\ 2 \\ -1 \end{pmatrix} = 0$ (M1) for setting equation

$(k)(k-2) + (-3)(2) + (k-2)(-1) = 0$ (A1) for correct approach

$k^2 - 2k - 6 - k + 2 = 0$

$k^2 - 3k - 4 = 0$

$(k+1)(k-4) = 0$

$k = -1$ or $k = 4$ A2

[4]

Exercise 31

1. Consider the lines L_1 and L_2 defined by $L_1 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix} + s \begin{pmatrix} k-1 \\ 20 \\ -10 \end{pmatrix}$ and

$L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} k+2 \\ k-2 \\ k \end{pmatrix}$, $k \in \mathbb{R}$. It is given that L_1 and L_2 are perpendicular. Find the possible values of k.

[4]

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2. Consider the lines L_1 and L_2 defined by $L_1 : \mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + s\begin{pmatrix} -4k \\ k \\ 0 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 10 \end{pmatrix} + t\begin{pmatrix} k \\ 1 \\ 1 \end{pmatrix}$, $k \in \mathbb{R}$. It is given that L_1 and L_2 are not perpendicular. Find the range of values of k.

[4]

3. Consider the lines L_1 and L_2 defined by $L_1 : \mathbf{r} = \begin{pmatrix} 7 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 15 \\ -8 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix}$.

It is given that L_1 and L_2 intersect at P.

(a) Find the value of s, the parameter value at P.

[2]

(b) Hence, find the coordinates of P.

[3]

4. Consider the lines L_1 and L_2 defined by $L_1 : \mathbf{r} = \begin{pmatrix} k \\ -5 \\ -4 \end{pmatrix} + s\begin{pmatrix} 3 \\ -4 \\ -3 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $k \in \mathbb{R}$. It is given that L_1 and L_2 intersect.

(a) Find the value of s, the parameter value at the point of intersection.

[3]

(b) Hence, find the value of k.

[2]

32 Paper 1 – Vector Equations of Lines

Example

The line L_1 passes through the point $\text{A}(3, 4, 6)$ and its direction vector is $\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$.

(a) Write down the vector equation of L_1.

[2]

(b) The vector $\mathbf{v} = \begin{pmatrix} k \\ -3 \\ 7 \end{pmatrix}$ is perpendicular to L_1.

(i) Find the value of k.

(ii) Hence, write down the vector equation of the line passing through A and is parallel to $\mathbf{v}$.

[4]

8

Solution

(a) $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ A2

[2]

(b) (i) $(1)(k)+(3)(-3)+(0)(7)=0$ (A1) for correct equation

$k-9=0$

$k=9$ A1

(ii) $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + s\begin{pmatrix} 9 \\ -3 \\ 7 \end{pmatrix}$ A2

[4]

Exercise 32

1. The line L_1 passes through the point $\mathrm{A}(1, -3, 5)$ and its direction vector is $\begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$.

(a) Write down the vector equation of L_1.

[2]

(b) Another point B is given by $\mathrm{B}(3, -8, 0)$.

(i) Write down $\overrightarrow{\mathrm{AB}}$.

(ii) Hence, write down the vector equation of the line passing through $\mathrm{C}(0, 3, 0)$ and is parallel to $\overrightarrow{\mathrm{AB}}$.

[3]

2. The line L_1 passes through the points $\mathrm{A}(1, 4, 7)$ and $\mathrm{B}(-2, -8, 5)$.

(a) (i) Write down $\overrightarrow{\mathrm{AB}}$.

(ii) Hence, write down the vector equation of the line passing through A and is parallel to $\overrightarrow{\mathrm{AB}}$.

[3]

$\mathrm{C}(k, 0, 4)$ is a point such that $\mathrm{BC} \perp \mathrm{AB}$.

(b) Find the value of k.

[3]

3. The line L_1 passes through the point $A(3, -2, -1)$ and its direction vector is $\begin{pmatrix} 5 \\ 4 \\ 2 \end{pmatrix}$. The vector equation of another line L_2 is $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$. $\mathbf{v}$ is a vector which is perpendicular to both L_1 and L_2.

(a) Find $\mathbf{v}$.

[3]

(b) Hence, write down the vector equation of the line passing through the point of intersection of L_1 and L_2, and is parallel to $\mathbf{v}$.

[2]

4. The line L_1 passes through the points $A(2, 0, 1)$ and $B(3, 3, 3)$. Another line L_2 passes through the points A and $C(4, -2, -2)$.

(a) Write down

(i) $\overrightarrow{AB}$;

(ii) $\overrightarrow{AC}$.

[2]

(b) Hence, find $\overrightarrow{AB} \times \overrightarrow{AC}$.

[2]

(c) Write down the vector equation of the line passing through C and is parallel to $\overrightarrow{AB} \times \overrightarrow{AC}$.

[2]

8

33 Paper 1 – Components of Vectors

Example

The vectors $\mathbf{a}$ and $\mathbf{b}$ are given as $\mathbf{a}=\begin{pmatrix}1\\3\\6\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-4\\-5\end{pmatrix}$.

(a) Write down $2\mathbf{a}+\mathbf{b}$.

[1]

(b) Hence, find the component of $2\mathbf{a}+\mathbf{b}$ parallel to $\mathbf{a}$.

[3]

Solution

(a) $2\mathbf{a}+\mathbf{b}=\begin{pmatrix}4\\2\\7\end{pmatrix}$ A1

[1]

(b) The required component

$=\frac{(2\mathbf{a}+\mathbf{b})\cdot\mathbf{a}}{|\mathbf{a}|}$ (M1) for valid approach

$=\frac{(4)(1)+(2)(3)+(7)(6)}{\sqrt{1^2+3^2+6^2}}$ (A1) for substitution

$=7.66698172$

$=7.67$ A1

[3]

Exercise 33

1. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given as $\mathbf{a}=\begin{pmatrix}5\\3\\0\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}0\\3\\0\end{pmatrix}$.

(a) Write down $\mathbf{a}+3\mathbf{b}$.

[1]

(b) Hence, find the component of $\mathbf{b}$ parallel to $\mathbf{a}+3\mathbf{b}$.

[3]

2. The vectors $\mathbf{a}$ and $\mathbf{b}$ are given as $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$.

(a) Write down $3\mathbf{a} - 2\mathbf{b}$.

[1]

(b) Hence, find the component of $3\mathbf{a} - 2\mathbf{b}$ perpendicular to $\mathbf{a}$.

[3]

3. The coordinates of two points A and B are $A(2, 6, 3)$ and $B(3, 4, -2)$. Let O be the origin.

(a) Find the component of $\vec{OA}$ parallel to $\vec{OB}$.

[3]

(b) (i) Find $\left|\vec{OA}\right|$.

(ii) Hence, find $A\hat{O}B$.

[4]

8

4. The coordinates of two points A and B are $A(3, 12, 4)$ and $B(5, 3, 1)$. Let O be the origin.

(a) Find the component of $\vec{OA}$ perpendicular to $\vec{OB}$.

[3]

(b) (i) Find $\left|\vec{OA}\right|$.

(ii) Hence, find $O\hat{A}B$.

[4]

34 Paper 2 – Applications in Kinematics

Example

A particle A is moving on a x-y plane. The displacement of A in the first 2 seconds is given by $\mathbf{s}_{\text{A}} = \begin{pmatrix} 3t \\ 12-t \end{pmatrix}$, where t is the time in seconds after the start of motion.

(a) Find the coordinates of A 2 seconds after the start of motion.

[2]

(b) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

[2]

A starts to accelerate such that the velocity of A after $t = 2$ is given by $\mathbf{v}_{\text{A}} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

(c) Find the coordinates of A

(i) 5 seconds after it accelerates;

(ii) 10 seconds after the start of motion.

[4]

(d) Find the acute angle between the line of motion of A and the x-axis.

[2]

A collides with the another particle B at $t = 10$. It is given that B travels at a constant velocity $\mathbf{v}_{\text{B}} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$.

(e) Find the coordinates of B 5 seconds after A accelerates.

[2]

(f) Hence, find the distance between the two particles 5 seconds after A accelerates.

[2]

Solution

(a) $\mathbf{s}_A = \begin{pmatrix} 3(2) \\ 12-2 \end{pmatrix}$ (M1) for substitution

$\mathbf{s}_A = \begin{pmatrix} 6 \\ 10 \end{pmatrix}$

Thus, the required coordinates are $(6, 10)$. A1

[2]

(b) $\mathbf{r} = \begin{pmatrix} 0 \\ 12 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ A2

[2]

(c) (i) $\begin{pmatrix} 6 \\ 10 \end{pmatrix} + 5\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 36 \\ 0 \end{pmatrix}$ (M1) for valid approach

Thus, the required coordinates are $(36, 0)$. A1

(ii) $\begin{pmatrix} 6 \\ 10 \end{pmatrix} + 8\begin{pmatrix} 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 54 \\ -6 \end{pmatrix}$ (M1) for valid approach

Thus, the required coordinates are $(54, -6)$. A1

[4]

(d) $\tan\theta = \frac{2}{6}$ (M1) for valid approach

$\theta = 18.43494882^\circ$

Thus, the required angle is 18.4°. A1

[2]

(e) $\begin{pmatrix} 54 \\ -6 \end{pmatrix} - 3\begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} 39 \\ -21 \end{pmatrix}$ (M1) for valid approach

Thus, the required coordinates are $(39, -21)$. A1

[2]

(f) The required distance

$= \sqrt{(39-36)^2 + (-21-0)^2}$ (A1) for substitution

$= 21.21320344$

$= 21.2$ A1

[2]

8

Exercise 34

1. A particle A is moving on a x - y plane. The displacement of A in the first 5 seconds is given by $\mathbf{s}_{\text{A}} = \begin{pmatrix} 10+t \\ 2t \end{pmatrix}$, where t is the time in seconds after the start of motion.

(a) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

[2]

(b) Find the coordinates of A 5 seconds after the start of motion.

[2]

A starts to change its velocity such that the velocity of A after $t = 5$ is given by $\mathbf{v}_{\text{A}} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$.

(c) Find the coordinates of A

(i) 7 seconds after it changes its velocity;

(ii) 20 seconds after the start of motion.

[4]

A collides with the another particle B at $t = 20$. It is given that B travels at a constant velocity $\mathbf{v}_{\text{B}} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

(d) Find the coordinates of B at $t = 0$.

[2]

(e) Hence, write down the vector equation for the displacement of B, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

[2]

(f) Find the distance between the two particles 8 seconds before they collide.

[3]

2. A ball A is moving on a smooth table which can be modelled by a x - y plane. Its initial position is $(3, 4)$. A collides with another stationary ball B after 1 second, and the initial position of B is $(15, 9)$.

(a) Find the initial distance between the two balls.

[2]

(b) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, where t is the time in seconds after the start of motion.

[2]

B is at $(15, 15)$ one second after colliding with A.

(c) Find the velocity vector of B.

[2]

A changes its velocity to $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ after the collision. It is given that A is at $(21, 3)$ x seconds after collision.

(d) Find x.

[2]

(e) Find the angle between the velocity vectors of A and B.

[3]

It is given that the horizontal edge of the table is given by the equation $y = 24$.

(f) Find the amount of time needed for B to reach the edge after A starts its motion.

[3]

3. Two planes A and B are going to land on an airport. The airport and the surrounding region can be modelled by a three-dimensional coordinate plane, where 1 unit on the coordinate axes represents 1 km. At 8:00, A is 8 km above the position $(-800, 160)$. It aims to land at the position $(0, 20, 0)$ at 10:00. Let t be the number of hours after 8:00.

8

(a) Find the velocity vector of A.

[2]

(b) Find the speed of A.

[2]

(c) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$,

[2]

The vector equation of B is given by $\mathbf{r} = \begin{pmatrix} -880 \\ -180 \\ 15 \end{pmatrix} + t\begin{pmatrix} 300 \\ 60 \\ -5 \end{pmatrix}$.

(d) Find the time when B lands on the ground.

[2]

(e) Hence, find the coordinates of B when it lands on the ground.

[2]

After A is landed on the airport at 10:00, an electric car is going to pick up passengers from A to the terminal. The velocity vector of the car is $\begin{pmatrix} 5s^2 - 2s \\ 4s - 10s^2 \end{pmatrix}$, where s is the number of hours after 10:00.

(f) Find the time for the car to stop again after 10:00.

[3]

4. In a game, two balls A and B are moving in a three-dimensional space, which can be modelled by a three-dimensional coordinate plane. At the start of the game, A is at $(20, -10, 0)$ and its velocity vector is $\begin{pmatrix} 5 \\ 5 \\ 0 \end{pmatrix}$.

(a) Write down the vector equation for the displacement of A, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$,

[2]

After p seconds, A is at $(40, 10, 0)$.

(b) Find p.

[2]

At the start of the game, B is at $(20, 20, 20)$. After 4 seconds, it is at $(40, 0, 0)$.

(c) Find the velocity vector of B.

[2]

(d) Write down the vector equation for the displacement of B, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$,

[2]

(e) Find the shortest distance between A and B.

[4]

(f) Hence, write down the time when A and B are closest to each other.

[1]

35 Paper 2 – Areas and Volumes

Example

The vector equations of the lines L_1 and L_2 are $\mathbf{r}=\begin{pmatrix}1\\3\\7\end{pmatrix}+t\begin{pmatrix}2\\-4\\3\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}9\\-7\\12\end{pmatrix}+s\begin{pmatrix}4\\-2\\-1\end{pmatrix}$ respectively. Let A and B be the points on L_1 and L_2 with parameters $t=0$ and $s=0$ respectively.

L_1 and L_2 intersect at C.

(a) (i) Find the value of s, the parameter value at C.

(ii) Hence, find the coordinates of C.

[5]

(b) Let θ be the acute angle between L_1 and L_2. Show that $\cos\theta=\frac{13}{\sqrt{609}}$.

[3]

(c) (i) Write down $\overrightarrow{CA}$.

(ii) Write down $\overrightarrow{CB}$.

(iii) Hence, find the area of the triangle ABC.

[5]

The coordinates of the point D are $(-7,-15,-2)$.

(d) Find the vector equation of the line L_3 that passes the point D and is parallel to $\overrightarrow{CA}\times\overrightarrow{CB}$.

[2]

8

Solution

(a) (i) $L_1:\begin{cases} x=1+2t \\ y=3-4t \\ z=7+3t \end{cases}$, $L_2:\begin{cases} x=9+4s \\ y=-7-2s \\ z=12-s \end{cases}$ (M1) for valid approach

$1+2t=9+4s$

$2t=8+4s$

$t=4+2s$

$3-4t=-7-2s$

$\therefore 3-4(4+2s)=-7-2s$ (M1) for substitution

$-13-8s=-7-2s$

$-6=6s$

$s=-1$ A1

(ii) $\begin{cases} x=9+4(-1)=5 \\ y=-7-2(-1)=-5 \\ z=12-(-1)=13 \end{cases}$ (M1) for substitution

Thus, the coordinates of C are $(5,-5,13)$. A1

[5]

(b) $\mathbf{b}_1=2\mathbf{i}-4\mathbf{j}+3\mathbf{k}$

$\mathbf{b}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$

$\mathbf{b}_1\cdot\mathbf{b}_2=|\mathbf{b}_1||\mathbf{b}_2|\cos\theta$ M1

$(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})$

$=(\sqrt{2^2+(-4)^2+3^2})(\sqrt{4^2+(-2)^2+(-1)^2})\cos\theta$ A1

$(2)(4)+(-4)(-2)+(3)(-1)=(\sqrt{29})(\sqrt{21})\cos\theta$ A1

$13=\sqrt{609}\cos\theta$

$\cos\theta=\dfrac{13}{\sqrt{609}}$ AG

[3]

(c) (i) $\overrightarrow{\text{CA}}=-4\mathbf{i}+8\mathbf{j}-6\mathbf{k}$ A1

(ii) $\overrightarrow{\text{CB}}=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$ A1

(iii) The area of the triangle ABC

$= \frac{1}{2}\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|$ (M1) for valid approach

$= \frac{1}{2}\left|(-4\mathbf{i} + 8\mathbf{j} - 6\mathbf{k}) \times (4\mathbf{i} - 2\mathbf{j} - \mathbf{k})\right|$

$= \frac{1}{2}\left|\begin{pmatrix} (8)(-1) - (-6)(-2) \\ (-6)(4) - (-4)(-1) \\ (-4)(-2) - (8)(4) \end{pmatrix}\right|$

$= \frac{1}{2}\left|-20\mathbf{i} - 28\mathbf{j} - 24\mathbf{k}\right|$

$= \frac{1}{2}\sqrt{(-20)^2 + (-28)^2 + (-24)^2}$ A1

$= 20.97617696$

$= 21.0$ A1

[5]

(d) The vector equation of the line L_3 :

$\mathbf{r} = \begin{pmatrix} -7 \\ -15 \\ -2 \end{pmatrix} + u\begin{pmatrix} -20 \\ -28 \\ -24 \end{pmatrix}$ A2

[2]

8

Exercise 35

1. The vector equations of the lines L_1 and L_2 are $\mathbf{r}=\begin{pmatrix}15\\11\\6\end{pmatrix}+t\begin{pmatrix}6\\3\\2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}0\\7\\-4\end{pmatrix}+s\begin{pmatrix}-3\\2\\-6\end{pmatrix}$ respectively. Let A and B be the points on L_1 and L_2 with parameters $t=0$ and $s=0$ respectively.

L_1 and L_2 intersect at C.

(a) (i) Find the value of t, the parameter value at C.

(ii) Hence, find the coordinates of C.

[5]

(b) (i) Write down $\overrightarrow{CA}$.

(ii) Write down $\overrightarrow{CB}$.

(iii) Hence, find the area of the triangle ABC.

[5]

(c) Find the vector equation of the line L_3 that passes the point C and is parallel to $\overrightarrow{CA}\times\overrightarrow{CB}$.

[2]

D is a point on L_3 such that the coordinates of D are $(73,-95,d)$.

(d) Find the value of d.

[2]

(e) (i) Write down $\overrightarrow{CD}$.

(ii) Hence, find the volume of the pyramid ABCD.

[4]

2. The vector equations of the lines L_1 and L_2 are $\mathbf{r}=\begin{pmatrix}8\\8\\7\end{pmatrix}+t\begin{pmatrix}2\\0\\0\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}6\\8+2\sqrt{3}\\7\end{pmatrix}+s\begin{pmatrix}1\\\sqrt{3}\\0\end{pmatrix}$ respectively. Let A and B be the points on L_1 and L_2 with parameters $t=-1$ and $s=-1$ respectively.

(a) Show that the coordinates of C, the point of intersection of L_1 and L_2, are $(4, 8, 7)$.

[3]

(b) Find the acute angle between L_2 and the y-axis.

[3]

(c) (i) Write down $\overrightarrow{CA}$.

(ii) Write down $\overrightarrow{CB}$, giving the answer in exact values.

(iii) Hence, find the value of $\hat{ACB}$.

[5]

(d) Find the area of the triangle ABC.

[3]

D, E and F are the points such that ABCFED is a prism, where the triangle ABC is congruent to the triangle DEF. It is given that the total surface area of the prism ABCFED is $2(30+\sqrt{3})$. Let h be the height of the prism ABCFED.

(e) (i) Find the value of h.

(ii) Hence, find the volume of the prism ABCFED.

[4]

8

3. The vector equations of the lines L_1 and L_2 are $\mathbf{r}=\begin{pmatrix}14\\18\\8\end{pmatrix}+t\begin{pmatrix}-5\\-6\\-2\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-6\\-14\\-6\end{pmatrix}+s\begin{pmatrix}1\\2\\1\end{pmatrix}$ respectively. L_1 and L_2 intersect at P.

(a) (i) Show that $t=2$, where t is the parameter value at P.

(ii) Hence, find the coordinates of P.

[5]

$\mathrm{R}(a, 4, 3)$ is a point lying on L_2.

(b) Write down the value of a.

[1]

Q is a point on L_1 such that $\overrightarrow{\mathrm{RQ}}=-4\mathbf{i}-4\mathbf{j}-\mathbf{k}$.

(c) (i) Write down $\overrightarrow{\mathrm{OR}}$.

(ii) Hence, find the coordinates of Q.

[3]

Let T and S be the points on L_1 and L_2 where $\overrightarrow{\mathrm{PT}}=6\overrightarrow{\mathrm{PQ}}$ and $\overrightarrow{\mathrm{PS}}=14\overrightarrow{\mathrm{PR}}$.

(d) Write down

(i) $\overrightarrow{\mathrm{PQ}}$;

(ii) $\overrightarrow{\mathrm{PT}}$;

(iii) $\overrightarrow{\mathrm{PR}}$;

(iv) $\overrightarrow{\mathrm{PS}}$.

(v) Hence, find the exact area of the quadrilateral QRST.

[8]

4. The coordinates of the points O, A, B, C and D are $(0, 0, 0)$, $(3, 0, 0)$, $(3, 0, 3)$, $(0, 0, 3)$ and $(0, -3, 0)$ respectively. E is a point on the line segment BD such that $\mathrm{AE} \perp \mathrm{BD}$.

(a) (i) Find the vector equation of BD, giving the answer in the form $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.

(ii) Express $\overrightarrow{\mathrm{AE}}$ in terms of t.

(iii) Hence, show that the coordinates of E are $(2, -1, 2)$.

[7]

(b) (i) Write down $\overrightarrow{\mathrm{BA}}$ and $\overrightarrow{\mathrm{BC}}$.

Let $\mathbf{n}_1 = \overrightarrow{\mathrm{BA}} \times \overrightarrow{\mathrm{BD}}$ and $\mathbf{n}_2 = \overrightarrow{\mathrm{BC}} \times \overrightarrow{\mathrm{BD}}$.

(ii) Find $\mathbf{n}_1$.

(iii) Find $\mathbf{n}_2$.

(iv) Hence, find the acute angle between $\mathbf{n}_1$ and $\mathbf{n}_2$.

[9]

F is a point on the y-axis such that the volume of the pyramid OABCF is 15.

(c) Find the possible coordinates of F.

[4]

Chapter

9

Graph Theory

SUMMARY POINTs

✓ Terminologies of graphs:
Vertex: A point on a graph
Edge: Arcs that connect vertices
Walk: A sequence of edges
Path: A sequence of edges that passes through any vertex and any edge at most once
Degree of a vertex: Number of edges connecting the vertex
Connected graph: A graph that there exists at least one walk between any two vertices
Unconnected graph: A graph that there exist at least two vertices that there is no walk between them
Subgraph of a graph: A collection of some edges and vertices of the original graph
Loop: An edge that starts and ends at the same vertex

SUMMARY POINTs

✓ Terminologies of graphs:

Simple graph: A graph that has no loops and no multiple edges connecting the same pair of vertices

Multiple graph: A graph that has multiple edges connecting at least one pair of vertices

Cycle: A path that the starting vertex is the end vertex

Tree: A connected graph with no cycles

Spanning tree: A tree that connects all vertices in the graph

✓ Directed graphs:

1. Directed graph: A graph that all edges are assigned with directions
2. In-degree of a vertex: Number of edges connecting and pointing towards the vertex
3. Out-degree of a vertex: Number of edges connecting and pointing away from the vertex

✓ Adjacency matrix $\mathbf{M}$ of a graph with n vertices:

1. $n \times n$: Order of $\mathbf{M}$
2. The entry $m_{ij} = 1$ if there is an edge connecting the vertex i and the vertex j, and $m_{ij} = 0$ if otherwise
3. $\mathbf{M}^p$ shows the number of walks of length p in the graph
4. $\sum_{r=1}^{p} \mathbf{M}^r$ shows the number of walks of length less than or equal to p in the graph
5. The column sum of a transition matrix of a directed graph must be equal to 1

✓ Algorithms of finding minimum spanning trees:

1. Kruskal's algorithm
2. Prim's algorithm

SUMMARY POINTs

✓ Eulerian trails and circuits:

1. Trail: A sequence of edges that passes through any edge at most once
2. Circuit: A trail that the starting vertex is the end vertex
3. Eulerian trail: A trail that passes through all edges of a graph
4. Eulerian circuit: A circuit that passes through all edges of a graph
5. An Eulerian trail exists if there exists two and only two vertices of odd degree
6. An Eulerian circuit exists if all vertices are of even degree
7. Chinese postman problem can be used to find the route of minimum weight that covers all edges of a graph

✓ Hamiltonian paths and cycles:

1. Complete graph: A graph that there exists an edge for any pair of two vertices
2. Hamiltonian path: A path that passes through all vertices of a graph
3. Hamiltonian cycle: A cycle that passes through all vertices of a graph

✓ Travelling Salesman problem:

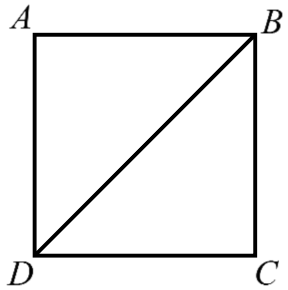
1. Travelling Salesman problem can be used to find the cycle of minimum weight that passes through all vertices of a graph
2. Nearest neighbour algorithm can be used to find the upper bound of the solution of a travelling salesman problem
3. Deleted vertex algorithm can be used to find the lower bound of the solution of a travelling salesman problem

Solutions of Chapter 9

36 Paper 1 – Adjacency Matrices

Example

Consider the following unweighted graph:



(a) Write down the degree of D. [1]

(b) Write down the adjacency matrix **M** of the graph. [2]

(c) Hence, find the number of walks of length 2 from A to C. [2]

9

Solution

(a) 3 A1

[1]

(b) $\mathbf{M}=\begin{pmatrix}0&1&0&1\\1&0&1&1\\0&1&0&1\\1&1&1&0\end{pmatrix}$ A2

[2]

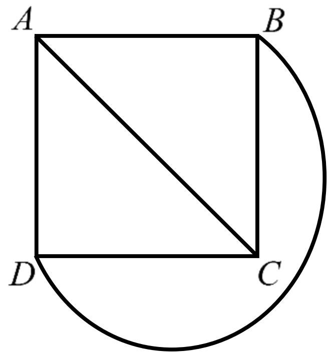
(c) $\mathbf{M}^2=\begin{pmatrix}2&1&2&1\\1&3&1&2\\2&1&2&1\\1&2&1&3\end{pmatrix}$ (M1) for valid approach

Thus, the number of walks of length 2 from A to C is 2. A1

[2]

Exercise 36

1. Consider the following unweighted graph:



(a) Write down the degree of B .

[1]

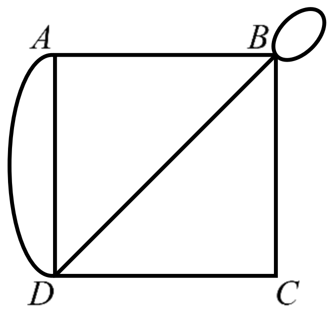
(b) Write down the adjacency matrix **M** of the graph.

[2]

(c) Hence, find the number of walks of length 3 from A to itself.

[2]

2. Consider the following unweighted graph:



(a) Write down the degree of B .

[1]

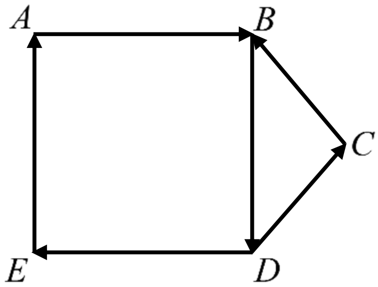
(b) Write down the adjacency matrix **M** of the graph.

[2]

(c) Hence, find the total number of walks of length 4 from each vertex to itself.

[3]

3. Consider the following directed graph:



(a) Write down the

(i) in-degree of B ;

(ii) out-degree of B .

[2]

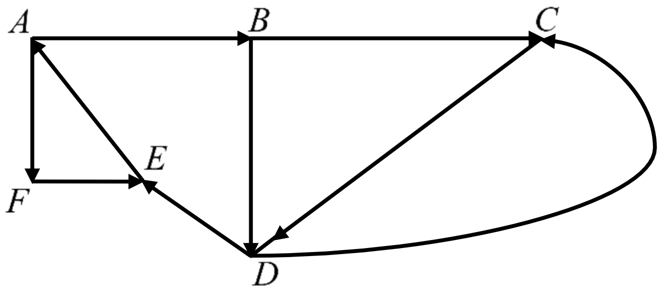
(b) Write down the adjacency matrix **M** of the graph.

[3]

(c) Hence, find the total number of walks of length 12 from D to B .

[2]

4. Consider the following directed graph:



(a) Write down the

(i) in-degree of A ;

(ii) out-degree of D .

[2]

(b) Write down the adjacency matrix **M** of the graph.

[3]

(c) Hence, find the total number of walks of length at most 3 from A to E .

[2]

37 Paper 1 – Kruskal's Algorithm

Example

The following cost adjacency matrix shows the information of a graph with five vertices A , B , C , D and E :

	A	B	C	D	E
A	-	30	26	48	52
B	30	-	40	26	22
C	26	40	-	24	32
D	48	26	24	-	40
E	52	22	32	40	-

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(a) State the edge of the smallest cost.

[1]

(b) By using the algorithm, find the minimum spanning tree.

[3]

(c) Write down the cost of the minimum spanning tree.

[1]

Solution

(a) BE A1

[1]

(b) For any two edges correct A1
For all edges correct A1
1. Choose BE of cost 22
2. Choose CD of cost 24
3. Choose AC of cost 26
4. Choose BD of cost 26

Thus, the minimum spanning tree is a tree containing BE, CD, AC and BD. A1

[3]

(c) 98 A1

[1]

Exercise 37

1. The following weight adjacency matrix shows the information of a graph with five vertices A, B, C, D and E:

	A	B	C	D	E
A	-	16	22	40	30
B	16	-	12	18	24
C	22	12	-	36	32
D	40	18	36	-	12
E	30	24	32	12	-

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(a) State the edge(s) of the smallest weight.

[1]

(b) By using the algorithm, find the minimum spanning tree.

[3]

(c) Write down the weight of the minimum spanning tree.

[1]

2. The following cost adjacency matrix shows the information of a graph with seven vertices A, B, C, D, E, F and G:

	A	B	C	D	E	F	G
A	-	33	x	51	72	36	54
B	33	-	66	70	45	64	76
C	x	66	-	30	39	80	56
D	51	70	30	-	78	62	72
E	72	45	39	78	-	90	30
F	36	64	80	62	90	-	58
G	54	76	56	72	30	58	-

Kruskal's algorithm is used to find the minimum spanning tree for this graph. The cost corresponding to the edge AC is x, where $x > 30$ and $x \in \mathbb{Z}$.

(a) State the edge(s) of the smallest cost.

[1]

(b) Write down the maximum possible value of x such that AC is the only sixth edge to be chosen in this algorithm.

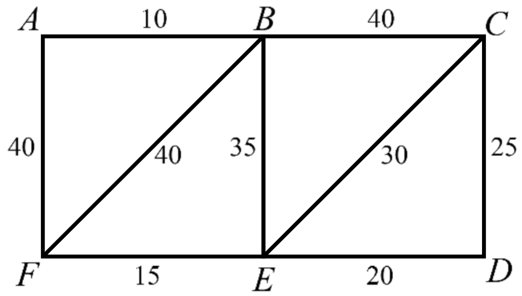
[1]

It is given that $x = 42$.

(c) By using the algorithm, find the minimum spanning tree. [3]

(d) Write down the cost of the minimum spanning tree. [1]

3. Consider the following weighted graph:



Kruskal's algorithm is used to find the minimum spanning tree for this graph.

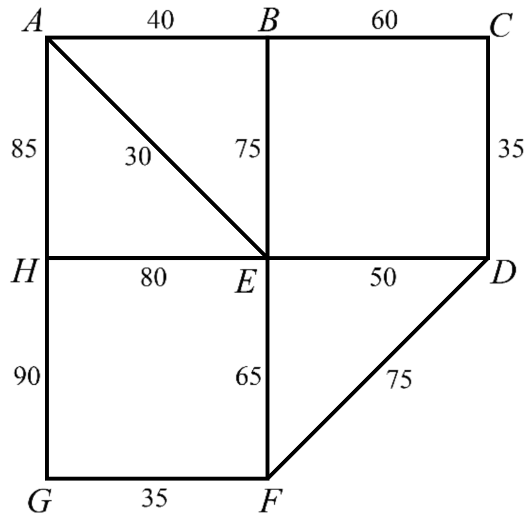
(a) Write down the adjacency matrix **M** of the graph. [2]

(b) State the edge of the smallest weight. [1]

(c) By using the algorithm, find the minimum spanning tree. [3]

(d) Write down the weight of the minimum spanning tree. [1]

4. Consider the following weighted graph:



Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(a) Write down the adjacency matrix **M** of the subgraph with five vertices D, E, F, G and H only. [2]

(b) State the edge of the smallest weight. [1]

(c) By using the algorithm, find the minimum spanning tree. [4]

(d) Write down the weight of the minimum spanning tree. [1]

9

38 Paper 1 – Prim's Algorithm

Example

The following cost adjacency matrix shows the information of a graph with five vertices A, B, C, D and E:

	A	B	C	D	E
A	-	40	27	58	59
B	40	-	43	46	28
C	27	43	-	44	37
D	58	46	44	-	41
E	59	28	37	41	-

Prim's algorithm, starting at C, is used to find the minimum spanning tree for this graph.

(a) State the edge of the smallest cost.

[1]

(b) By using the algorithm, find the minimum spanning tree.

[3]

(c) Write down the cost of the minimum spanning tree.

[1]

Solution

(a) AC A1

[1]

(b) For any two edges correct A1
For all edges correct A1
1. Choose AC of cost 27
2. Choose CE of cost 37
3. Choose BE of cost 28
4. Choose DE of cost 41

Thus, the minimum spanning tree is a tree containing AC, CE, BE and DE. A1

[3]

(c) 133 A1

[1]

Exercise 38

1. The following weight adjacency matrix shows the information of a graph with five vertices A , B , C , D and E :

	A	B	C	D	E
A	-	76	32	43	31
B	76	-	52	12	22
C	32	52	-	31	33
D	43	12	31	-	14
E	31	22	33	14	-

Prim's algorithm, starting at A , is used to find the minimum spanning tree for this graph.

(a) State the edge of the greatest weight. [1]

(b) By using the algorithm, find the minimum spanning tree. [3]

(c) Write down the weight of the minimum spanning tree. [1]

2. The following cost adjacency matrix shows the information of a graph with seven vertices A , B , C , D , E , F and G :

	A	B	C	D	E	F	G
A	-	38	89	63	70	26	51
B	38	-	56	82	x	44	73
C	89	56	-	42	30	40	55
D	63	82	42	-	70	52	62
E	70	x	30	70	-	60	40
F	26	44	40	52	60	-	28
G	51	73	55	62	40	28	-

Prim's algorithm, starting at B , is used to find the minimum spanning tree for this graph. The cost corresponding to the edge BE is x , where $x \in \mathbb{Z}$.

(a) Write down the maximum possible value of x such that BE is the first edge to be chosen in this algorithm. [1]

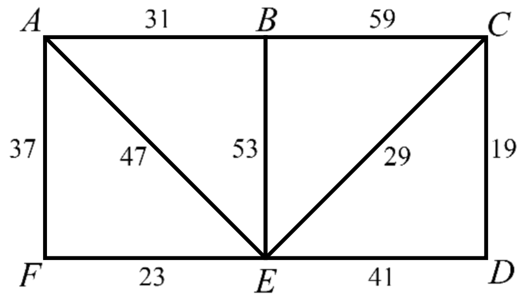
It is given that $x = 29$.

(b) By using the algorithm, find the minimum spanning tree. [3]

(c) Write down the cost of the minimum spanning tree. [1]

9

3. Consider the following weighted graph:



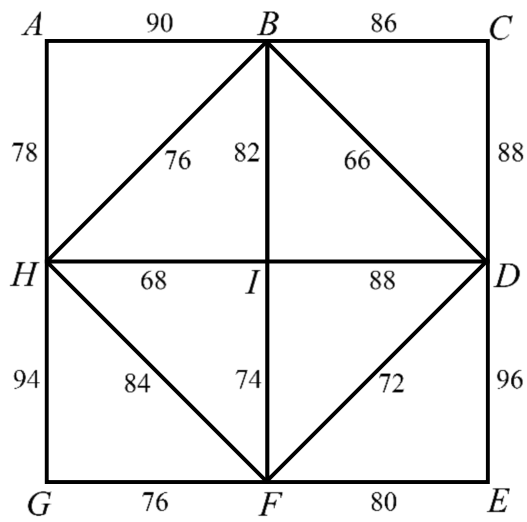
Prim's algorithm, starting at F, is used to find the minimum spanning tree for this graph.

(a) Write down the adjacency matrix **M** of the graph. [2]

(b) By using the algorithm, find the minimum spanning tree. [3]

(c) Write down the weight of the minimum spanning tree. [1]

4. Consider the following weighted graph:



Prim's algorithm, starting at I, is used to find the minimum spanning tree for this graph.

(a) Write down the adjacency matrix **M** of the subgraph with five vertices D, F, G, H and I only. [2]

(b) State the edge of the greatest weight. [1]

(c) By using the algorithm, find the minimum spanning tree. [4]

(d) Write down the weight of the minimum spanning tree. [1]

39 Paper 2 – Chinese Postman Problems

Example

The following table shows the weight of the edges of a graph with six vertices A, B, C, D, E and F:

	A	B	C	D	E	F
A	-	22	-	-	23	27
B	22	-	18	12	20	-
C	-	18	-	18	-	-
D	-	12	18	-	15	-
E	23	20	-	15	-	5
F	27	-	-	-	5	-

(a) Draw a weighted graph that represents the above table.

[3]

(b) Write down

(i) the degree of A ;

(ii) the number of vertices of odd degree.

[2]

(c) Write down the adjacency matrix **M** of the graph.

[2]

(d) Hence, find the total number of walks of length 2 from E to A .

[2]

(e) Use the Chinese postman algorithm to find a possible route of minimum weight that passes through all edges, starting and finishing at A .

[3]

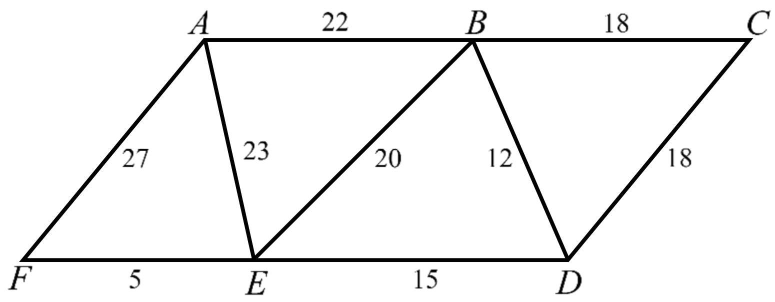
(f) Write down the corresponding weight of the route.

[1]

Solution

(a) For correct edges A1
For correct vertices A1
For correct weights A1

[3]



(b) (i) 3 A1

(ii) 2 A1

[2]

(c) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ A2

[2]

(d) $\mathbf{M}^2 = \begin{pmatrix} 3 & 1 & 1 & 2 & 2 & 1 \\ 1 & 4 & 1 & 2 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 3 & 1 & 1 \\ 2 & 2 & 2 & 1 & 4 & 1 \\ 1 & 2 & 0 & 1 & 1 & 2 \end{pmatrix}$ (M1) for valid approach

Thus, the total number of walks of length 2 from E to A is 2. A1

[2]

(e) For any three edges correct A1

For any six edges correct A1

1. Choose AB of weight 22
2. Choose BC of weight 18
3. Choose CD of weight 18
4. Choose DE of weight 15
5. Choose EF of weight 5
6. Choose FA of weight 27
7. Choose AE of weight 23
8. Choose EB of weight 20
9. Choose BD of weight 12
10. Choose DB of weight 12
11. Choose BA of weight 22

Thus, a possible route contains AB, BC, CD, DE, EF, FA, AE, EB, BD, DB and BA. A1

[3]

(f) 194 A1

[1]

Exercise 39

1. The following table shows the weight of the edges of a graph with six vertices A, B, C, D, E and F:

	A	B	C	D	E	F
A	-	20	55	-	40	25
B	20	-	10	-	50	-
C	55	10	-	25	40	-
D	-	-	25	-	30	-
E	40	50	40	30	-	10
F	25	-	-	-	10	-

(a) Draw a weighted graph that represents the above table.

[3]

(b) Write down

(i) the degree of E;

(ii) the number of vertices of odd degree.

[2]

(c) Write down the adjacency matrix **M** of the graph.

[2]

9

(d) Hence, find the total number of walks of length 4 from C to A .

[2]

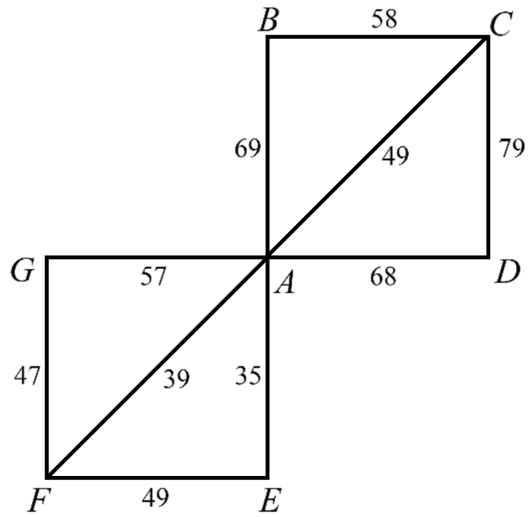
(e) Use the Chinese postman algorithm to find a possible route of minimum weight that passes through all edges, starting and finishing at E .

[3]

(f) Write down the corresponding weight of the route.

[1]

2. Consider the following weighted graph:



(a) Write down

(i) the degree of A ;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree.

[3]

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(b) State the edge of the smallest weight.

[1]

(c) By using the algorithm, find the minimum spanning tree.

[3]

(d) Write down the weight of the minimum spanning tree.

[1]

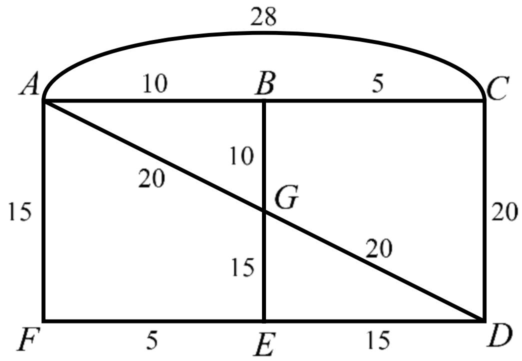
(e) Use the Chinese postman algorithm to find a possible route of minimum weight that passes through all edges, starting and finishing at F .

[3]

(f) Write down the corresponding weight of the route.

[1]

3. The following weighted graph shows a network of roads connecting seven cities A, B, C, D, E, F and G. The weight on each edge shows the time, in minutes, needed to travel along the road represented by that edge.



(a) Write down

(i) the degree of D;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree;

(iv) the minimum amount of time needed to travel from A to C.

[4]

(b) Use the Chinese postman algorithm to find a possible route of minimum time required to passes through all roads, starting and finishing at C.

[3]

(c) Write down the corresponding time required of the route.

[1]

Assume that it is not necessary to start and finish at the same point. A car starts its journey at C.

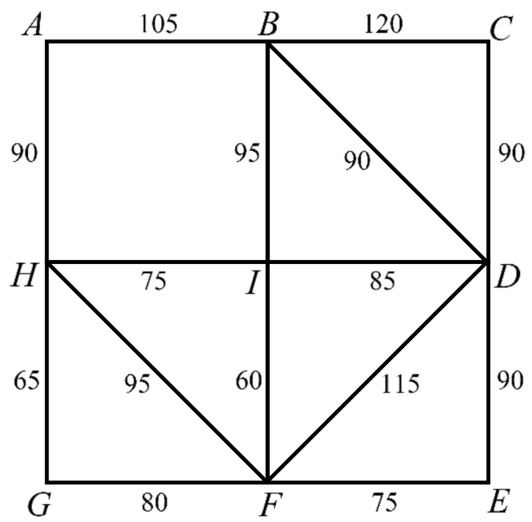
(d) (i) Write down a finishing point other than C such that the total time required to passes through all roads attains its minimum.

(ii) Hence, use the Chinese postman algorithm to find a possible route of minimum time required to passes through all roads.

(iii) Write down the corresponding time required of the route.

[5]

4. The following weighted graph shows a network of roads connecting nine fountains A, B, C, D, E, F, G, H and I, in a park. The weight on each edge shows the time, in seconds, needed to travel along the road represented by that edge.



(a) Write down

(i) the degree of F;

(ii) the number of vertices of odd degree;

(iii) the number of vertices of even degree;

(iv) the minimum amount of time needed to travel from B to H.

[4]

(b) Use the Chinese postman algorithm to find a possible route of minimum time required to passes through all roads, starting at D and finishing at F.

[3]

(c) Write down the corresponding time required of the route.

[1]

Assume that it is necessary to start and finish at the same point. Peter starts his journey at D.

(d) (i) Use the Chinese postman algorithm to find a possible route of minimum time required to passes through all roads.

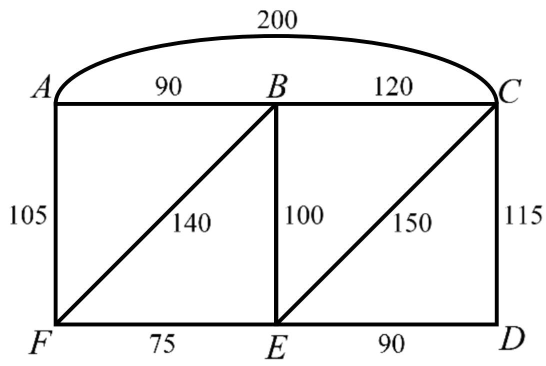
(ii) Write down the corresponding time required of the route.

[4]

40 Paper 2 – Travelling Salesman Problems

Example

Consider the following weighted graph:



(a) Write down

(i) the degree of B ;

(ii) the vertices of odd degree.

[2]

(b) State the edge of the smallest weight.

[1]

(c) Is Eulerian circuit exists in the above graph? Explain your answer.

[2]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F
A	-	90	200	270	p	105
B	90	-	120	190	100	q
C	200	120	-	115	150	225
D	270	190	115	-	90	165
E	p	100	150	90	-	75
F	105	q	225	165	75	-

(d) Write down the value of

(i) p ;

(ii) q .

[2]

(e) Using the nearest neighbour algorithm, starting and finishing at F, find an upper bound of the total weight of a cycle that passes through all six vertices.

[3]

(f) Using the deleted vertex algorithm by deleting the vertex F, find a lower bound of the total weight of a cycle that passes through all six vertices.

[4]

Solution

(a) (i) 4 A1

(ii) A, F A1

[2]

(b) EF A1

[1]

(c) Eulerian circuit does not exist. A1
As not all vertices are of even degree. A1

[2]

(d) (i) 180 A1

(ii) 140 A1

[2]

(e) For any three edges correct A1
For all edges correct A1

1. Choose FE of weight 75
2. Choose ED of weight 90
3. Choose DC of weight 115
4. Choose CB of weight 120
5. Choose BA of weight 90
6. Choose AF of weight 105

Thus, the required upper bound is 595. A1

[3]

(f) For any two edges correct A1
For all edges correct A1

1. Choose AB of weight 90
2. Choose DE of weight 90
3. Choose BE of weight 100
4. Choose CD of weight 115

Therefore, the weight of a minimum spanning tree after deleting the vertex F is 395. A1

The required lower bound

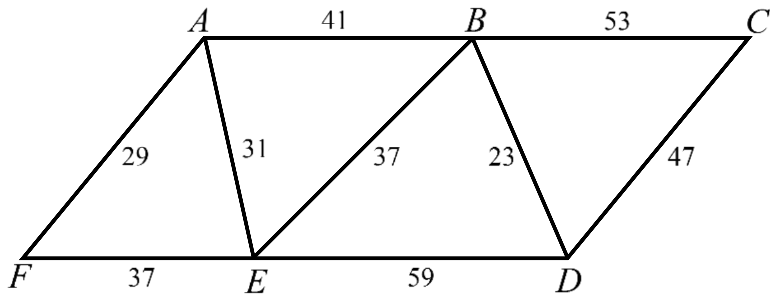
$= 395 + 75 + 105$

$= 575$ A1

[4]

Exercise 40

1. Consider the following weighted graph:



(a) Write down

(i) the degree of E;

(ii) the vertices of odd degree.

[2]

(b) State the edge of the smallest weight.

[1]

(c) Is Eulerian trail exists in the above graph? Explain your answer.

[2]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F
A	-	41	94	64	31	29
B	41	-	53	23	37	p
C	94	53	-	47	q	123
D	64	23	47	-	59	96
E	31	37	q	59	-	37
F	29	p	123	96	37	-

(d) Write down the value of

(i) p;

(ii) q.

[2]

(e) Using the nearest neighbour algorithm, starting and finishing at B, find an upper bound of the total weight of a cycle that passes through all six vertices.

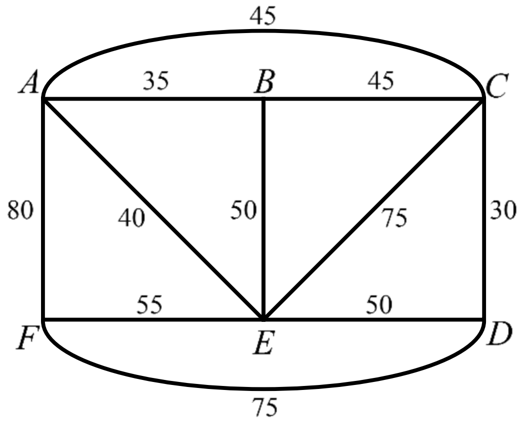
[3]

(f) Using the deleted vertex algorithm by deleting the vertex B, find a lower bound of the total weight of a cycle that passes through all six vertices.

[4]

9

2. Consider the following weighted graph:



(a) Is Eulerian trail exists in the above graph? Explain your answer.
[2]

(b) Write down the adjacency matrix $\mathbf{M}$ of the graph.
[2]

(c) Hence, find the total number of walks of length 3 from C to A .
[2]

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(d) By using the algorithm, find the minimum spanning tree.
[3]

(e) Write down the weight of the minimum spanning tree.
[1]

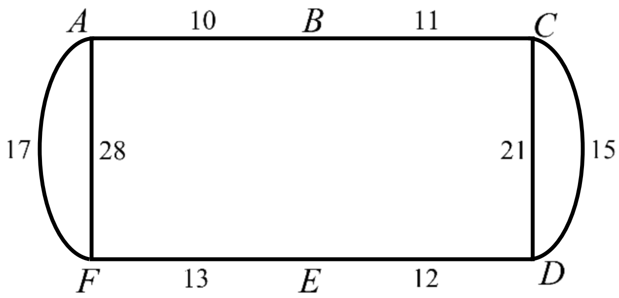
The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F
A	-	35	45	75	40	80
B	35	-	45	75	50	105
C	45	45	-	30	75	130
D	75	75	30	-	50	75
E	40	50	75	50	-	55
F	80	105	130	75	55	-

(f) Using the nearest neighbour algorithm, starting and finishing at D , find an upper bound of the total weight of a cycle that passes through all six vertices.
[3]

(g) Using the deleted vertex algorithm by deleting the vertex D , find a lower bound of the total weight of a cycle that passes through all six vertices.
[4]

3. Consider the following weighted graph:



(a) Write down

(i) the number of vertices of odd degree;

(ii) the number of vertices of even degree.

[2]

(b) Use the Chinese postman algorithm to find a possible route of minimum weight that passes through all edges, starting and finishing at C .

[3]

(c) Write down the corresponding weight of the route.

[1]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F
A	-	10	21	36	30	17
B	10	-	11	26	44	27
C	21	11	-	15	27	46
D	36	26	15	-	12	25
E	30	44	27	12	-	13
F	17	27	46	25	13	-

(d) Using the nearest neighbour algorithm, starting and finishing at A , find an upper bound of the total weight of a cycle that passes through all six vertices.

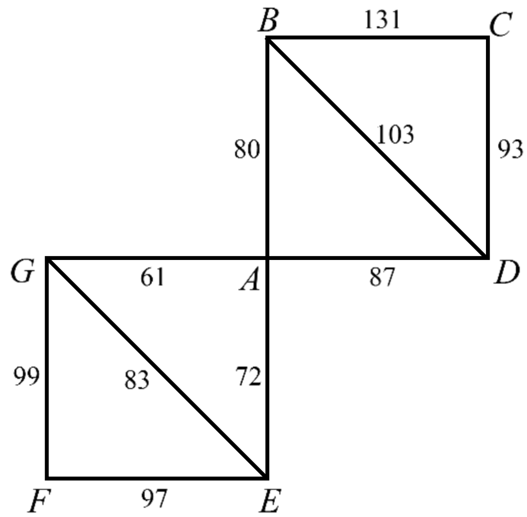
[3]

(e) Using the deleted vertex algorithm by deleting the vertex A , find a lower bound of the total weight of a cycle that passes through all six vertices.

[4]

9

4. Consider the following weighted graph:



(a) Is Eulerian circuit exists in the above graph? Explain your answer.

[2]

Prim's algorithm, starting at A, is used to find the minimum spanning tree for this graph.

(b) State the edge of the least weight.

[1]

(c) By using the algorithm, find the minimum spanning tree.

[3]

(d) Write down the weight of the minimum spanning tree.

[1]

The following table shows the least weight of a path connecting any two vertices.

	A	B	C	D	E	F	G
A	-	80	180	87	72	160	61
B	80	-	131	103	152	240	141
C	180	131	-	93	252	340	241
D	87	103	93	-	159	247	148
E	72	152	252	159	-	97	83
F	160	240	340	247	97	-	99
G	61	141	241	148	83	99	-

(e) Using the nearest neighbour algorithm, starting and finishing at D, find an upper bound of the total weight of a cycle that passes through all seven vertices.

[3]

(f) Using the deleted vertex algorithm by deleting the vertex D, find a lower bound of the total weight of a cycle that passes through all seven vertices.

[4]

Chapter

10

Differentiation

SUMMARY POINTs

✓ Derivatives of a function $y = f(x)$:

1. $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$: Second derivative

2. $\frac{d^n y}{dx^n} = f^{(n)}(x)$: n-th derivative

✓ Rules of differentiation:

<table>
<tr><td>$f(x) = \sin x \Rightarrow f'(x) = \cos x$</td><td>$f(x) = p(q(x)) \Rightarrow f'(x) = p'(q(x)) \cdot q'(x)$</td></tr>
<tr><td>$f(x) = \cos x \Rightarrow f'(x) = -\sin x$</td><td rowspan="2">$f(x) = p(x)q(x)$
$\Rightarrow f'(x) = p'(x)q(x) + p(x)q'(x)$</td></tr>
<tr><td>$f(x) = \tan x \Rightarrow f'(x) = \frac{1}{\cos^2 x}$</td></tr>
<tr><td>$f(x) = e^x \Rightarrow f'(x) = e^x$</td><td rowspan="2">$f(x) = \frac{p(x)}{q(x)}$
$\Rightarrow f'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{(q(x))^2}$</td></tr>
<tr><td>$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$</td></tr>
</table>

SUMMARY POINTs

✓ $f''(a)=0$ and $f''(x)$ changes sign at $x=a$: $(a, f(a))$ is a point of inflexion of $f(x)$

✓ $\frac{dN}{dt}=\frac{dN}{dx}\cdot\frac{dx}{dt}$: Rate of change of N with respect to the time t

✓ Tests for optimization:

1. First derivative test
2. Second derivative test

✓ Applications in kinematics:

1. $s(t)$: Displacement with respect to the time t
2. $v(t)=s'(t)$: Velocity
3. $a(t)=v'(t)$: Acceleration

Solutions of Chapter 10

41 Paper 1 – Gradients of Tangents and Normals

Example

Let $f(x) = e^x \sin x$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $x = \frac{\pi}{2}$.

(ii) Hence, find the gradient of the normal to the curve of f at $x = \frac{\pi}{2}$.

[4]

Solution

(a) $f'(x) = (e^x)(\sin x) + (e^x)(\cos x)$ (M1) for product rule

$f'(x) = e^x(\sin x + \cos x)$ A1

[2]

(b) (i) The gradient of the tangent

$= f'\left(\frac{\pi}{2}\right)$

$= e^{\frac{\pi}{2}}\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)$ (M1) for substitution

$= 4.810477381$

$= 4.81$ A1

(ii) The gradient of the normal

$= \frac{-1}{f'\left(\frac{\pi}{2}\right)}$

$= \frac{-1}{4.810477381}$ (M1) for valid approach

$= -0.2078795764$

$= -0.208$ A1

[4]

10

Exercise 41

1. Let $f(x) = 3x\cos x$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $x = \frac{3\pi}{2}$.

(ii) Hence, find the gradient of the normal to the curve of f at $x = \frac{3\pi}{2}$.

[4]

2. Let $f(x) = e^{-3x}$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $x = 0.1$.

(ii) Hence, find the gradient of the normal to the curve of f at $x = 0.1$.

[4]

3. Let $f(x) = \cos(x^2)$.

(a) Find $f'(x)$.

[2]

The gradient of the tangent to the curve of f at $x = a$ is $-2a$, where $1 < a < 2$.

(b) Find a.

[3]

4. Let $g(x) = x^2 \ln x$.

(a) Find $g'(x)$.

[2]

The gradient of the tangent to the curve of g at $x = a$ is $\sqrt{a}$, $a > 0$.

(b) Find a.

[3]

42 Paper 1 – Equations of Tangents and Normals

Example

Let $f(x) = 2e^{-x}$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $(0, 2)$.

(ii) Hence, find the equation of the tangent to the curve of f at $(0, 2)$, giving the answer in the form $y = ax + b$.

[5]

Solution

(a) $f'(x) = 2(e^{-x})(-1)$ (M1) for chain rule

$f'(x) = -2e^{-x}$ A1

[2]

(b) (i) The gradient of the tangent

$= f'(0)$

$= -2e^{-0}$ (M1) for substitution

$= -2$ A1

(ii) The equation of the tangent:

$y = -2x + b$ (M1) for setting equation

$2 = -2(0) + b$ (M1) for substitution

$b = 2$

$\therefore y = -2x + 2$ A1

[5]

Exercise 42

1. Let $f(x) = \sin 3x$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $(\pi, 0)$.

(ii) Hence, find the equation of the tangent to the curve of f at $(\pi, 0)$, giving the answer in the form $y = ax + b$ and in terms of π.

[5]

2. Let $f(x) = \ln x^3$.

(a) Find $f'(x)$.

[2]

(b) (i) Find the gradient of the tangent to the curve of f at $(1, 0)$.

(ii) Hence, write down the gradient of the normal to the curve of f at $(1, 0)$.

(iii) Find the equation of the normal to the curve of f at $(1, 0)$, giving the answer in the form $y = ax + b$.

[6]

3. Let $f(x) = e^{3x}$.

(a) Find $f'(x)$.

[2]

(b) Hence, write down the gradient of the tangent to the curve of f at (k, e^{3k}), giving the answer in terms of k.

[1]

It is given that the equation of the tangent to the curve of f at (k, e^{3k}), where $0 < k < 10$, is $3x - \frac{1}{e^3}y - 2 = 0$.

(c) Find the value of k.

[3]

4. Let $f(x)=\ln\sqrt{x}$.

(a) Find $f'(x)$.

[2]

(b) Hence, write down the gradient of the tangent to the curve of f at $(2, \ln\sqrt{2})$.

[1]

It is given that the equation of the normal to the curve of f at $(2, \ln\sqrt{2})$ is $y=mx+(\ln\sqrt{2}-2m)$, where m is a constant.

(c) Find the exact value of the y-intercept of the normal to the curve of f at $(2, \ln\sqrt{2})$.

[3]

43 Paper 1 – Second Derivative Test

Example

Let $f(x)=x^3+6x^2-15x$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

(c) (i) Write down $f''(x)$.

(ii) Hence, determine the x-coordinate of the local maximum of f.

(iii) Write down the y-coordinate of the local maximum of f.

[4]

Solution

(a) $f'(x)=3x^2+6(2x)-15(1)$ (A1) for correct approach

$f'(x)=3x^2+12x-15$ A1

[2]

(b) $f'(x)=0$

$3x^2+12x-15=0$

$3(x+5)(x-1)=0$ (A1) for factorization

$x=-5$ or $x=1$ A1

[2]

(c) (i) $f''(x)=6x+12$ A1

(ii) $f''(-5)=6(-5)+12$

$f''(-5)=-18<0$ R1

Therefore, f attains its local maximum at $x=-5$.

Thus, the x-coordinate of the local maximum of f is -5. A1

(iii) 100 A1

[4]

Exercise 43

1. Let $f(x)=1+9x+3x^2-x^3$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

(c) (i) Write down $f''(x)$.

(ii) Hence, determine the x-coordinate of the local minimum of f.

(iii) Write down the y-coordinate of the local minimum of f.

[4]

2. Let $f(x)=e^{-\frac{1}{2}x^2}$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

It is given that $f''(x)=(x^2-1)e^{-\frac{1}{2}x^2}$.

(c) (i) Show that the x-coordinate of the local maximum of f is 0.

(ii) Hence, write down the y-coordinate of the local maximum of f.

[3]

3. Let $f(x)=\dfrac{4x}{(x+1)^2}$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

It is given that $f''(x)=\dfrac{8(x-2)}{(x+1)^4}$.

(c) (i) Show that the x-coordinate of the local maximum of f is 1.

(ii) Hence, write down the y-coordinate of the local maximum of f.

[3]

4. Let $f(x)=\sin\left(2x-\dfrac{\pi}{3}\right)-3$ for $-\dfrac{\pi}{3}\le x\le\dfrac{2\pi}{3}$.

(a) Find $f'(x)$.

[2]

(b) Hence, solve the equation $f'(x)=0$.

[2]

(c) (i) Write down $f''(x)$.

(ii) Hence, determine the x-coordinate of the local minimum of f.

(iii) Write down the y-coordinate of the local minimum of f.

[4]

44 Paper 1 – Rate of Change

Example

The number of bacteria, n, in a dish, after t minutes is given by $n = 1200e^{0.08t}$.

(a) Find the value of n when $t = 0$.

[2]

(b) (i) Find $\frac{dn}{dt}$.

(ii) Hence, write down the rate at which n is increasing when $t = 24$.

[3]

(c) After k minutes, the rate of increase in n is greater than 2000 bacteria per minute. Find the least value of k, where $k \in \mathbb{Z}$.

[3]

Solution

(a) $n(0) = 1200e^{0.21(0)}$ (M1) for substitution

$n(0) = 1200$ A1

[2]

(b) (i) $\frac{dn}{dt} = 1200(e^{0.08t})(0.08)$ (M1) for chain rule

$\frac{dn}{dt} = 96e^{0.08t}$ A1

(ii) $\left.\frac{dn}{dt}\right|_{t=24} = 654.81201 \text{ min}^{-1}$

$\left.\frac{dn}{dt}\right|_{t=24} = 655 \text{ min}^{-1}$ A1

[3]

(c) $\left.\frac{dn}{dt}\right|_{t=k} > 2000$

$96e^{0.08k} > 2000$ (M1) for setting inequality

$96e^{0.08k} - 2000 > 0$ (M1) for valid approach

By considering the graph of $y = 96e^{0.08k} - 2000$,

$k > 37.956928$.

Thus, the least value of k is 38. A1

[3]

10

Exercise 44

1. The number of insects, n, after t months is given by $n = 300e^{0.28t}$.

(a) Find the value of n when $t = 0$.

[2]

(b) (i) Find $\frac{dn}{dt}$.

(ii) Hence, write down the rate at which n is increasing when $t = 6$.

[3]

(c) After k months, the rate of increase in n is greater than 1000 insects per month. Find the least value of k, where $k \in \mathbb{Z}$.

[3]

2. The volume of an object, V cm^3, t seconds after the start of an experiment is given by $V = \sqrt{100 - t^2}$, for $0 \le t \le 10$.

(a) Find the value of V when $t = 8$.

[2]

(b) (i) Find $\frac{dV}{dt}$.

(ii) Hence, write down the rate at which V is decreasing when $t = 6$.

[3]

(c) Within k seconds after the start of the experiment, the rate of decrease in V is smaller than $0.9\text{ cm}^3\text{s}^{-1}$. Find the greatest value of k, where $k \in \mathbb{Z}$.

[3]

3. The population of sheep in a farm, p, t months after 1 January 2019 is given by $p = 250\sin(2t + 3.9) + 750$.

(a) Find the number of sheep in the farm on 1 June 2019.

[3]

(b) Find $\frac{dp}{dt}$.

[2]

After n days, where $n \in \mathbb{Z}$, the rate of increase in p is equal to 0 per month for the first time.

(c) Find the value of n.

[3]

4. The population of wolf in a forest, w, t months after 1 April 2008 is given by $w = 145\cos(0.5t - 5.2) + 1020$.

(a) Find the number of wolves in the forest on 1 May 2009, correct the answer to the nearest integer.

[3]

(b) Find $\frac{dw}{dt}$.

[2]

Assume that there are 30 days per month. After n days, where $n \in \mathbb{Z}$, the rate of increase in w attains its maximum for the first time.

(c) Find the value of n.

[3]

45 Paper 1 – Problems in Kinematics

Example

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t)=t^3\sin t$, $0\le t\le 3$.

(a) Find the maximum distance of the particle from O.

[2]

(b) (i) Find $s'(t)$.

(ii) Find the time when the particle first changes direction.

(iii) Hence, write down the acceleration of the particle at the instant it first changes direction.

[5]

Solution

(a) By considering the graph of $y=t^3\sin t$, the maximum distance $=9.3794925\text{ cm}$ $=9.38\text{ cm}$ — (M1) for valid approach; A1

[2]

(b) (i) $s'(t)=(3t^2)(\sin t)+(t^3)(\cos t)$ — (M1) for product rule

$s'(t)=3t^2\sin t+t^3\cos t$ — A1

(ii) By considering the graph of $y=3t^2\sin t+t^3\cos t$, the particle first changes direction at 2.4556439 s. — (M1) for valid approach

Thus, the required time is 2.46 s. — A1

(iii) $s''(2.4556439)=-28.04457\text{ cms}^{-2}$

$s''(2.4556439)=-28.0\text{ cms}^{-2}$ — A1

[5]

Exercise 45

1. The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t$, $0 \le t \le 2.5$.

(a) Find the maximum distance of the particle from O.

[2]

(b) (i) Find $s'(t)$.

(ii) Find the time when the particle first changes direction.

(iii) Hence, write down the acceleration of the particle at the instant it first changes direction.

[5]

2. The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = 4t \cos t$, $0 \le t \le 2.25$.

(a) Find the maximum distance of the particle from O.

[2]

(b) (i) Find the time when the particle first goes back to O.

(ii) Find $s'(t)$.

(iii) Hence, write down the acceleration of the particle at the instant it first goes back to O.

[5]

3. The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t + \sin(e^t)$, $0 \le t \le 2.8$.

(a) Find $v(t)$, the velocity of the particle at time t seconds.

[3]

(b) (i) Find the time when the particle changes direction for the 4th time.

(ii) Hence, write down the acceleration of the particle at the instant it changes direction for the 4th time.

[3]

4. The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = e^t \sin t$, $0 \le t \le 9.5$.

(a) Find the amount of time between the particle changes direction for the 1st time and the 3rd time.

[3]

(b) (i) Find the time when the particle is at the maximum distance from O.

(ii) Find $s'(t)$.

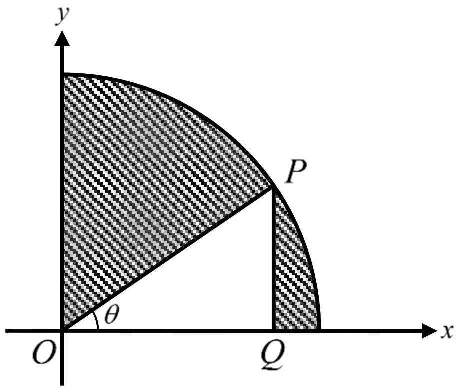
(iii) Hence, write down the acceleration of the particle at the instant when it is at the maximum distance from O.

[5]

46 Paper 2 – Optimization Problems

Example

The following diagram shows a quarter of a circle with centre O and radius 2. Let P be a point on the circumference and Q be a point on the x-axis such that PQ and OQ are perpendicular, with $P\hat{O}Q = \theta$ radians, where $0 \le \theta \le \frac{\pi}{2}$. Let S be the area of the shaded segment.



(a) Find the area of the triangle OPQ in terms of θ.

[3]

(b) Express S in terms of π and θ.

[2]

(c) (i) Find $\frac{dS}{d\theta}$.

(ii) Solve the equation $\frac{dS}{d\theta} = 0$.

It is given that $\frac{d^2S}{d\theta^2} = 8\sin\theta\cos\theta$.

(iii) Hence, find the value of θ when S is at a local minimum, justifying that it is a minimum.

(iv) Using (b) and (c)(iii) to find the minimum value of S.

[8]

(d) Write down the two values of θ for which S has its greatest value.

[2]

10

Solution

(a) The area of the triangle OPQ

$=\frac{1}{2}(OQ)(PQ)$ (M1) for valid approach

$=\frac{1}{2}(2\cos\theta)(2\sin\theta)$ (M1) for substitution

$=2\sin\theta\cos\theta$ A1

[3]

(b) $S=\frac{1}{4}\pi(2)^2-2\sin\theta\cos\theta$ (M1) for valid approach

$S=\pi-2\sin\theta\cos\theta$ A1

[2]

(c) (i) $\frac{dS}{d\theta}=0-2((\cos\theta)(\cos\theta)+(\sin\theta)(-\sin\theta))$ (M1) for product rule

$\frac{dS}{d\theta}=-2\cos^2\theta+2\sin^2\theta$ A1

(ii) $\frac{dS}{d\theta}=0$

$\therefore -2\cos^2\theta+2\sin^2\theta=0$

By considering the graph of

$y=-2\cos^2\theta+2\sin^2\theta$, $\theta=0.7853981634$. (M1) for valid approach

$\therefore \theta=0.785$ rad A1

(iii) $\left.\frac{d^2S}{d\theta^2}\right|_{\theta=0.7853981634}$

$=8(\sin 0.7853981634)(\cos 0.7853981634)$

$=4>0$ R1

Thus, S attains its minimum at

$\theta=0.785$ rad. A1

(iv) The minimum value of S

$=\pi-2\begin{pmatrix}\sin 0.7853981634\\ \cdot\cos 0.7853981634\end{pmatrix}$ (M1) for substitution

$=2.141592654$

$=2.14$ A1

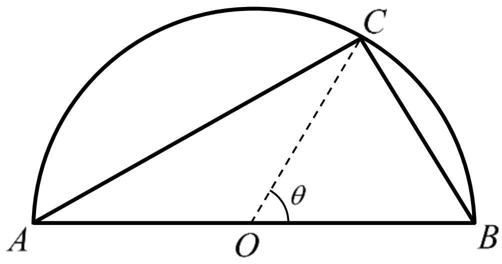
[8]

(d) $\theta=0$, $\theta=\frac{\pi}{2}$ A2

[2]

Exercise 46

1. The following diagram shows a semicircle with centre O, diameter AB and radius 4. Let C be a point on the circumference with $B\hat{O}C = \theta$ radians, where $0 \le \theta \le \pi$. Let P be the area of the triangle ABC.



(a) (i) Write down the value of $A\hat{O}C$.

(ii) Hence, by using $\sin(\pi - \theta) = \sin\theta$, express P in terms of θ.

[5]

(b) (i) Write down $\frac{dP}{d\theta}$.

(ii) Solve the equation $\frac{dP}{d\theta} = 0$.

(iii) Hence, find the value of θ when P is at a local maximum, justifying that it is a maximum.

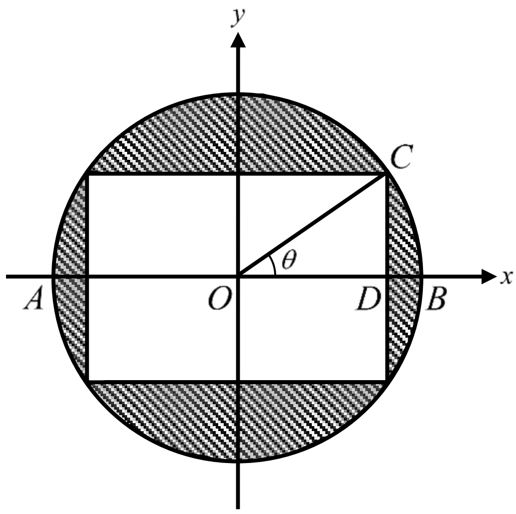
(iv) Using (a)(ii) and (b)(iii) to find the maximum value of P.

[8]

(c) Write down the two values of θ for which P has its least value.

[2]

2. The following diagram shows a circle with centre O, diameter AB and radius 10. Let C be a point on the circumference with $\text{B}\hat{\text{O}}\text{C} = \theta$ radians, where $0 \le \theta \le \frac{\pi}{2}$. D is a point on the x-axis such that CD and OD are perpendicular. A rectangle is inscribed in the circle. Let P be the total area of the shaded regions.



(a) (i) Express CD and OD in terms of θ.

(ii) Express the area of the inscribed rectangle in terms of θ.

(iii) Hence, express P in terms of π and θ.

[6]

(b) (i) Find $\frac{\text{d}P}{\text{d}\theta}$.

(ii) Solve the equation $\frac{\text{d}P}{\text{d}\theta} = 0$.

It is given that $\frac{\text{d}^2P}{\text{d}\theta^2} = 1600\sin\theta\cos\theta$.

(iii) Hence, find the value of θ when P is at a local minimum, justifying that it is a minimum.

(iv) Using (a) and (b)(iii) to find the minimum value of P.

[8]

(c) Write down the two values of θ for which P has its greatest value.

[2]

3. The temperature Q of a substance at time t is given by $Q(t)=t^3-12t^2+36t$, where $t\geq 0$.

(a) Find the t-intercepts of Q.

[3]

(b) (i) Find $Q'(t)$.

(ii) Solve the equation $Q'(t)=0$.

(iii) Hence, find the value of θ when Q is at a local maximum, justifying that it is a maximum.

(iv) Write down the maximum value of Q.

[8]

Let $R(t)=Q(t)-20$.

(c) (i) Find the minimum value of R.

(ii) Write down the corresponding value(s) of t.

[3]

4. The price P of a share at time t is given by $P(t)=-t^3+9t^2-24t+720$, where $0\leq t\leq 12$.

(a) Find the t-intercept of P.

[2]

(b) (i) Find $P'(t)$.

(ii) Solve the equation $P'(t)=0$.

(iii) Hence, find the value of θ when P is at a local minimum, justifying that it is a minimum.

(iv) Write down the minimum value of P.

[8]

It is given that $P(0)=720$.

(c) (i) Write down $P(4)$.

(ii) Hence, write down the maximum value of P.

[2]

Let $Q(t)=P(t-3)$.

(d) Find the corresponding value of t when Q attains its maximum.

[2]

Chapter 11

Integration and Trapezoidal Rule

SUMMARY POINTs

✓ Rules of integration:

$\int \sin x\,\mathrm{d}x = -\cos x + C$	$\int e^x\,\mathrm{d}x = e^x + C$
$\int \cos x\,\mathrm{d}x = \sin x + C$	$\int \frac{1}{x}\,\mathrm{d}x = \ln x + C$
$\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \tan x + C$	Integration by substitution
$\int_a^b f'(x)\,\mathrm{d}x = [f(x)]_a^b = f(b) - f(a)$	

✓ Areas on x - y plane, between $x = a$ and $x = b$:

1. $\int_a^b |f(x)|\,\mathrm{d}x$: Area between the graph of $f(x)$ and the x-axis
2. $\int_a^b |f(x) - g(x)|\,\mathrm{d}x$: Area between the graph of $f(x)$ and the graph of $g(x)$

SUMMARY POINTs

✓ Areas on x - y plane, between $y=c$ and $y=d$:

1. $\int_c^d |g(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the y-axis
2. $\int_c^d |g(y)-f(y)|\mathrm{d}y$: Area between the graph of $g(y)$ and the graph of $f(y)$

✓ Volumes of revolutions about the x-axis, between $x=a$ and $x=b$:

1. $V=\pi\int_a^b (f(x))^2\mathrm{d}x$: Volume of revolution when the region between the graph of $f(x)$ and the x-axis is rotated 360° about the x-axis
2. $V=\pi\int_a^b ((f(x))^2-(g(x))^2)\mathrm{d}x$: Volume of revolution when the region between the graphs of $f(x)$ and $g(x)$ is rotated 360° about the x-axis

✓ Volumes of revolutions about the y-axis, between $y=c$ and $y=d$:

1. $V=\pi\int_c^d (g(y))^2\mathrm{d}y$: Volume of revolution when the region between the graph of $g(y)$ and the y-axis is rotated 360° about the y-axis
2. $V=\pi\int_c^d ((g(y))^2-(f(y))^2)\mathrm{d}y$: Volume of revolution when the region between the graphs of $g(y)$ and $f(y)$ is rotated 360° about the y-axis

✓ Applications in kinematics:

1. $a(t)$: Acceleration with respect to the time t
2. $v(t)=\int a(t)\mathrm{d}t$: Velocity
3. $s(t)=\int v(t)\mathrm{d}t$: Displacement
4. $d=\int_{t_1}^{t_2}|v(t)|\mathrm{d}t$: Total distance travelled between t_1 and t_2

Solutions of Chapter 11

47 Paper 1 – Finding Original Functions

Example

Let $f'(x)=\dfrac{4x}{(2x^2+1)^6}$.

(a) Express $f(x)$ as an indefinite integral of x.

[1]

(b) Using the substitution $u=2x^2+1$ to integrate the integral in (a), giving the answer in terms of x.

[3]

It is given that $f(0)=-\dfrac{6}{5}$.

(c) Find the expression of $f(x)$.

[2]

Solution

(a) $f(x)=\displaystyle\int\frac{4x}{(2x^2+1)^6}\,\mathrm{d}x$ A1

[1]

(b) Let $u=2x^2+1$.

$\dfrac{\mathrm{d}u}{\mathrm{d}x}=4x\Rightarrow \mathrm{d}u=4x\mathrm{d}x$ A1

$\therefore f(x)=\displaystyle\int\frac{1}{u^6}\,\mathrm{d}u$ (A1) for substitution

$f(x)=\dfrac{1}{-5}u^{-5}+C$

$f(x)=-\dfrac{1}{5(2x^2+1)^5}+C$ A1

[3]

(c) $-\dfrac{6}{5}=-\dfrac{1}{5(2(0)^2+1)^5}+C$ (M1) for substitution

$C=-1$

$\therefore f(x)=-\dfrac{1}{5(2x^2+1)^5}-1$ A1

[2]

Exercise 47

1. Let $f'(x) = \cos^3 2x \sin 2x$.

(a) Express $f(x)$ as an indefinite integral of x.

[1]

(b) Using the substitution $u = \cos 2x$ to integrate the integral in (a), giving the answer in terms of x.

[3]

It is given that $f\left(\frac{\pi}{2}\right) = 3$.

(c) Find the expression of $f(x)$.

[2]

2. Let $f'(x) = 2x\sin(x^2)$.

(a) Express $f(x)$ as an indefinite integral of x.

[1]

(b) Using the substitution $u = x^2$ to integrate the integral in (a), giving the answer in terms of x.

[3]

It is given that $f(0) = -1$.

(c) Find the expression of $f(x)$.

[2]

3. Let $f'(x) = 3x^2(x^3+1)^6$.

(a) Express $f(x)$ as an indefinite integral of x.

[1]

(b) Using a suitable substitution to integrate the integral in (a), giving the answer in terms of x.

[4]

It is given that $f(-1) = 2$.

(c) Find the expression of $f(x)$.

[2]

4. Let $f'(x) = 4x^3e^{x^4}$.

(a) Using a suitable substitution to integrate $f'(x)$, giving the answer in terms of x .

[5]

It is given that $f(2) = e^{16} - 1$.

(b) Find the expression of $f(x)$.

[2]

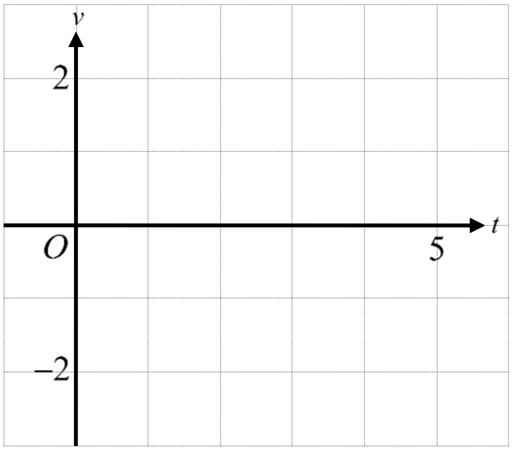
48 Paper 1 – Graphs in Kinematics

Example

The velocity of a particle in ms^{-1} is given by $v = e^{\cos t} - 1$, for $0 \le t \le 4$.

(a) On the grid below, sketch the graph of v.

[3]



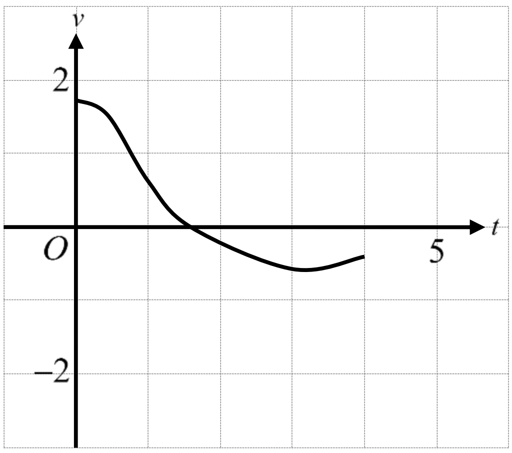
(b) Find the total distance travelled by the particle in the first four seconds.

[3]

Solution

(a) For approximately correct shape A1
For approximately correct x-intercept between 1 and 2 A1
For approximately correct endpoints and minimum point A1

[3]



(b) The total distance travelled

$= \int_0^4 |v(t)| \, dt$ (M1) for valid approach

$= \int_0^4 |e^{\cos t} - 1| \, dt$ (A1) for substitution

$= 2.732255679$

$= 2.73 \text{ m}$ A1

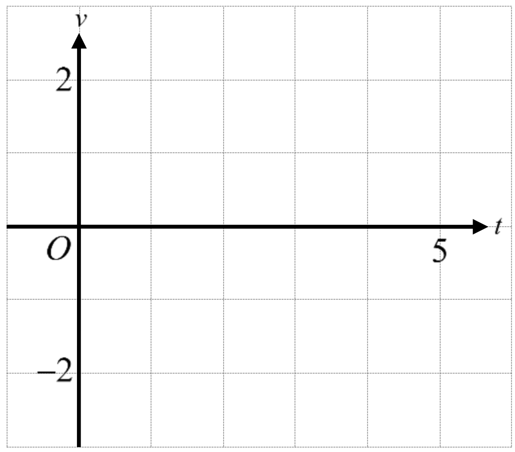
[3]

Exercise 48

1. The velocity of a particle in ms^{-1} is given by $v = \ln(t + 0.3)$, for $0 \le t \le 5$.

(a) On the grid below, sketch the graph of v.

[3]



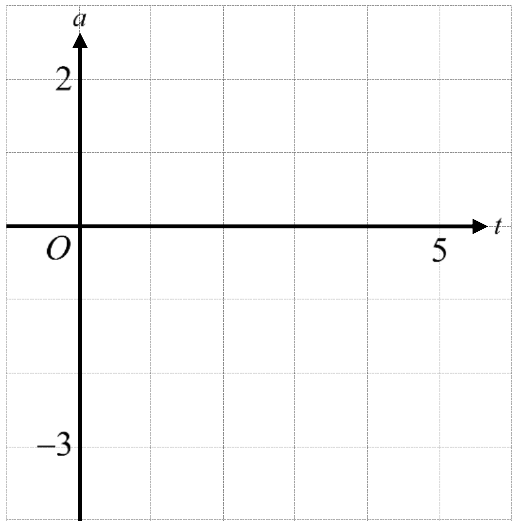
(b) Find the total distance travelled by the particle in the first five seconds.

[3]

2. The acceleration of a particle in ms^{-2} is given by $a = -(t-2)^2 + 1.9$, for $0 \le t \le 4$. The particle is at rest when $t = 0$.

(a) On the grid below, sketch the graph of a.

[3]



(b) Find $v(4) - v(0)$, where v is the velocity of the particle.

[3]

3. The acceleration of a particle in ms^{-2} is given by $a(t) = \dfrac{2t}{t^2+1}$, for $0 \le t \le 6$. The particle is at rest when $t = 0$.

(a) Using the substitution $u = t^2 + 1$ to find the expression of $v(t)$, the velocity of the particle, giving the answer in the form $f(t) + C$.

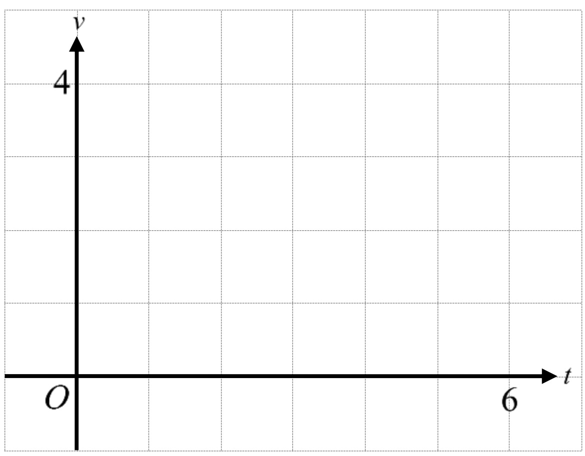
[4]

(b) By finding C, find the expression of $v(t)$.

[2]

(c) On the grid below, sketch the graph of $v(t)$.

[2]



4. The velocity of a particle in ms^{-1} is given by $v(t) = 2t\cos(t^2)$, for $0 \le t \le 3$. The displacement of the particle is 1 m when $t = 0$.

(a) Using the substitution $u = \sin(t^2)$ to find the expression of $s(t)$, the displacement of the particle, giving the answer in the form $f(t) + C$.

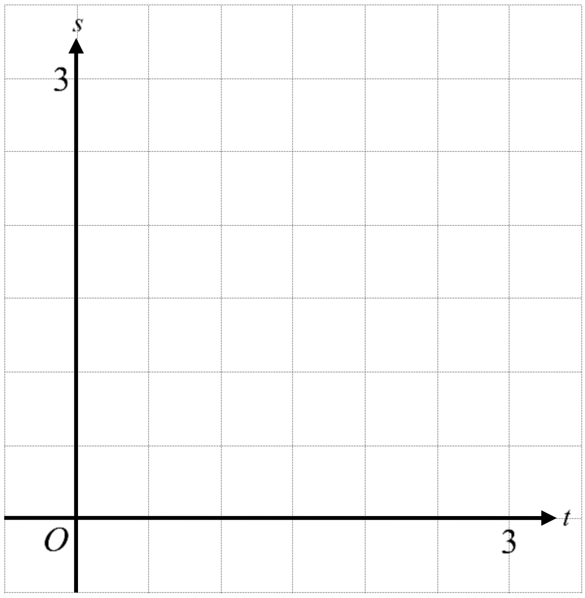
[3]

(b) By finding C, find the expression of $s(t)$.

[2]

(c) On the grid below, sketch the graph of s.

[3]



49 Paper 1 – Area Under a Curve

Example

Consider the graph of $f(x)=2\sin 2x-1$, where $0<x<\pi$.

(a) Find the x-intercept(s) of $f(x)$ in $0<x<\pi$.

[2]

(b) Hence, find the area of the region enclosed by the graph of f and the x-axis.

[2]

Solution

(a) By considering the graph of $y=2\sin 2x-1$,

$x=0.2617994$ or $x=1.3089969$. (M1) for valid approach

Thus, the x-intercepts are 0.262 and 1.31. A1

[2]

(b) The area of the region

$=\int_{0.2617994}^{1.3089969}\left|2\sin 2x-1\right|\mathrm{d}x$ (A1) for correct approach

$=0.6848532654$

$=0.685$ A1

[2]

Exercise 49

1. Consider the graph of $f(x)=-\cos \pi x$, where $0\le x\le 2$.

(a) Find the x-intercept(s) of $f(x)$ in $0\le x\le 2$.

[2]

(b) Hence, find the area of the region enclosed by the graph of f and the x-axis.

[2]

2. Consider the graph of $f(y)=e^{y-1}-1$, where $0\le y\le 2$.

(a) Find the y-intercept(s) of $f(y)$ in $0\le y\le 2$.

[2]

(b) Hence, find the area of the region enclosed by the graph of f and the coordinate axes.

[2]

3. Consider the graph of $f(y) = 3e^{-y^2} - 2$.

(a) Find the y-intercept(s) of $f(y)$.

[2]

(b) Hence, find the area of the region enclosed by the graph of f and the y-axis.

[2]

4. Consider the graph of $f(y) = y^3 - 21y^2 + 138y - 280$.

(a) Find the y-intercept(s) of $f(y)$.

[3]

(b) Hence, find the total area of the regions enclosed by the graph of f and the y-axis.

[2]

50 Paper 1 – Volumes of Revolutions

Example

R is defined to be the region bounded by the curve $y = \frac{1}{\sin \frac{\pi}{2} x}$ and the lines $x = \frac{1}{3}$, $x = 1$ and the x-axis.

(a) Find the area of R.

[2]

R is rotated through 2π radians about the x-axis to form a solid.

(b) Find the volume of the solid generated.

[2]

Solution

(a) The area of R

$= \int_{\frac{1}{3}}^{1} \left| \frac{1}{\sin \frac{\pi}{2} x} \right| \mathrm{d}x$ (A1) for correct approach

$= 0.8384014366$

$= 0.838$ A1

[2]

(b) The volume of the solid

$= \int_{\frac{1}{3}}^{1} \pi \left(\frac{1}{\sin \frac{\pi}{2} x} \right)^2 \mathrm{d}x$ (A1) for correct approach

$= 3.464101615$

$= 3.46$ A1

[2]

Exercise 50

1. R is defined to be the region bounded by the curve $y=\dfrac{\sqrt{\cos\left(x-\frac{\pi}{2}\right)}}{\sin\left(x-\frac{\pi}{2}\right)}$ and the lines $x=\dfrac{2\pi}{3}$, $x=\dfrac{5\pi}{6}$ and the x-axis.

(a) Find the area of R.

[2]

R is rotated through 2π radians about the x-axis to form a solid.

(b) Find the volume of the solid generated.

[2]

2. R is defined to be the region bounded by the curve $x=\pi e^{\pi y}$ and the lines $y=0.75$, the x-axis and the y-axis.

(a) Find the area of R.

[2]

R is rotated through 2π radians about the y-axis to form a solid.

(b) Find the volume of the solid generated.

[2]

3. R is defined to be the region bounded by the curve $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

(a) Write down the x-intercepts of the curve $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

[2]

(b) Make y^2 the subject of $\dfrac{x^2}{9}+\dfrac{y^2}{4}=1$.

[1]

A rugby model is formed by rotating the region R through 2π about the x-axis.

(c) Find the volume of the rugby model.

[2]

4. R is defined to be the region bounded by the curve $2x+y-6=0$ and the lines $y=4$, the x-axis and the y-axis

(a) Make x the subject of $2x+y-6=0$.

[1]

(b) Find the area of R.

[2]

A conical frustum is formed by rotating the region R through 2π about the y-axis.

(c) Find the volume of the conical frustum.

[2]

51 Paper 1 – Composite Figures and Objects

Example

The region R is enclosed by the x-axis, the graph of $y=\frac{3}{4}x-\frac{3}{2}$ and the graph of $y=\frac{3}{8}x^2-\frac{27}{4}x+30$. The domain of $y=\frac{3}{8}x^2-\frac{27}{4}x+30$ is $\{x : x \le 9\}$.

(a) Write down the x-intercept of

(i) $y=\frac{3}{4}x-\frac{3}{2}$;

(ii) $y=\frac{3}{8}x^2-\frac{27}{4}x+30$.

[2]

(b) Write down the coordinates of the point of intersection of the graph of $y=\frac{3}{4}x-\frac{3}{2}$ and the graph of $y=\frac{3}{8}x^2-\frac{27}{4}x+30$.

[2]

(c) Hence, find the area of R.

[3]

Solution

(a) (i) 2 A1

(ii) 8 A1

[2]

(b) (6, 3) A2

[2]

(c) The area of R

$=\int_2^6\left(\frac{3}{4}x-\frac{3}{2}\right)dx+\int_6^8\left(\frac{3}{8}x^2-\frac{27}{4}x+30\right)dx$ (A2) for correct approach

$=8.5$ A1

[3]

Exercise 51

1. The region R is enclosed by the x-axis, the graph of $y=e^{2x}-1$ and the graph of $y=e^{6}+2-x$.

(a) Write down the x-intercept of

(i) $y=e^{2x}-1$;

(ii) $y=e^{6}+2-x$, giving the answer in terms of e.

[2]

(b) Write down the x-coordinate of the point of intersection of the graph of $y=e^{2x}-1$ and the graph of $y=e^{6}+2-x$.

[1]

(c) Hence, find the area of R.

[3]

2. The region R is enclosed by the x-axis, the graph of $y=2\ln x$ and the graph of $y=8-\frac{4}{e^{2}}x$.

(a) Write down the x-intercept of

(i) $y=2\ln x$;

(ii) $y=8-\frac{4}{e^{2}}x$, giving the answer in terms of e.

[2]

(b) Write down the x-coordinate of the point of intersection of the graph of $y=2\ln x$ and the graph of $y=8-\frac{4}{e^{2}}x$.

[1]

(c) Hence, find the volume of the solid of revolution formed when R is rotated through 2π about the x-axis.

[3]

3. The region R is enclosed by the y-axis , the graph of $x = 3y - 3$ and the graph of $x = -3 + \ln\left|\frac{1}{1+y}\right|$.

(a) Write down the y-intercept of

(i) $x = 3y - 3$;

(ii) $x = -3 + \ln\left|\frac{1}{1+y}\right|$.

[2]

(b) Write down the y-coordinate of the point of intersection of the graph of $x = 3y - 3$ and the graph of $x = -3 + \ln\left|\frac{1}{1+y}\right|$.

[1]

(c) Hence, find the area of R.

[3]

4. The region R is enclosed by the y-axis , the graph of $y = \frac{1}{2}x + 1$ and the graph of $x = 4 - \sqrt{y-3}$. The range of $y = \frac{1}{2}x + 1$ is $\{y : y \le 3\}$.

(a) Write down the y-intercept of

(i) $y = \frac{1}{2}x + 1$;

(ii) $x = 4 - \sqrt{y-3}$.

[2]

(b) Make x the subject of $y = \frac{1}{2}x + 1$.

[1]

(c) Write down the y-coordinate of the point of intersection of the graph of $y = \frac{1}{2}x + 1$ and the graph of $x = 4 - \sqrt{y-3}$.

[1]

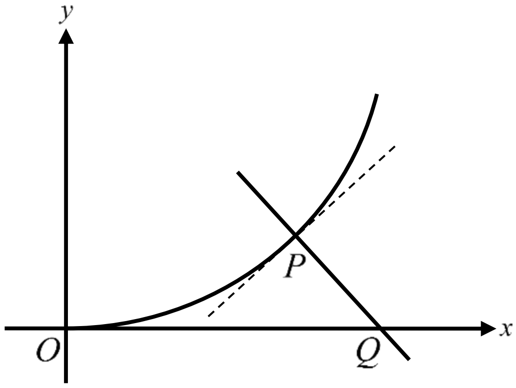
(d) Hence, find the volume of the solid of revolution formed when R is rotated through 2π about the y-axis.

[3]

52 Paper 2 – More about Areas

Example

Let $f(x)=x^3$. The following diagram shows part of the graph of f.



The point $\mathrm{P}(a, f(a))$, where $a>0$, lies on the graph of f. The normal at P crosses the x-axis at the point $\mathrm{Q}(b, 0)$.

It is given that the gradient of [PQ] is $-\frac{1}{3}$.

(a) (i) Find $f'(a)$.

(ii) Hence, find a.

(iii) Find b.

(iv) Find the equation of [PQ] in the form $y=mx+c$.

[9]

(b) Find the area between the graph of f, the normal at P and the x-axis.

[4]

Solution

(a) (i) $f'(a)\times -\frac{1}{3}=-1$ (M1) for valid approach

$f'(a)=3$ A1

(ii) $f'(x)=3x^2$ (A1) for correct approach

$f'(a)=3a^2$

$\therefore 3a^2=3$ (M1) for setting equation

$a^2=1$

$a=1$ A1

(iii) $\frac{1^3-0}{1-b}=-\frac{1}{3}$ (M1) for setting equation

$\frac{1}{1-b}=-\frac{1}{3}$

$-3=1-b$

$b=4$ A1

(iv) The equation of [PQ]:

$y=-\frac{1}{3}x+c$

$0=-\frac{1}{3}(4)+c$ (M1) for substitution

$c=\frac{4}{3}$

$\therefore y=-\frac{1}{3}x+\frac{4}{3}$ A1

[9]

(b) The required area

$=\int_0^1 f(x)\mathrm{d}x+\frac{(4-1)(1-0)}{2}$ M1A1

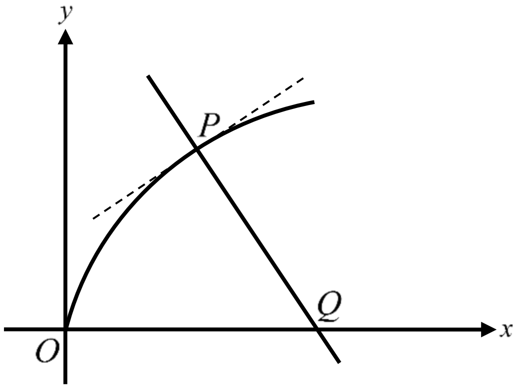
$=\int_0^1 x^3\mathrm{d}x+\frac{(3)(1)}{2}$ (A1) for correct approach

$=\frac{7}{4}$ A1

[4]

Exercise 52

1. Let $f(x) = x^{\frac{1}{2}}$. The following diagram shows part of the graph of f.



The point $P(a, f(a))$, where $a > 0$, lies on the graph of f. The normal at P crosses the x-axis at the point $Q(b, 0)$.

It is given that the gradient of [PQ] is -4.

(a) (i) Find $f'(a)$.

(ii) Hence, find a.

(iii) Find b.

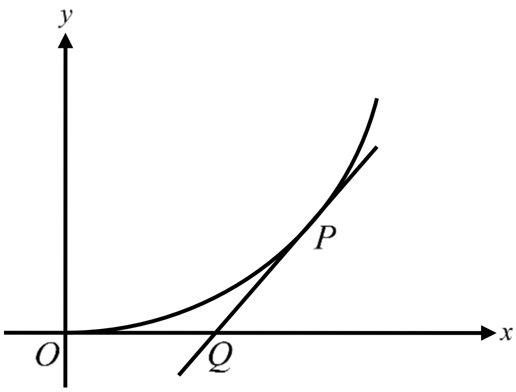
(iv) Find the equation of [PQ] in the form $y = mx + c$.

[9]

(b) Find the area between the graph of f, the normal at P and the x-axis.

[4]

2. Let $f(x) = x^2$. The following diagram shows part of the graph of f.



The point $P(a, f(a))$, where $a > 0$, lies on the graph of f. The tangent at P crosses the x-axis at the point $Q(1, 0)$.

(a) (i) Find two different expressions of $f'(a)$.

(ii) Hence, find a.

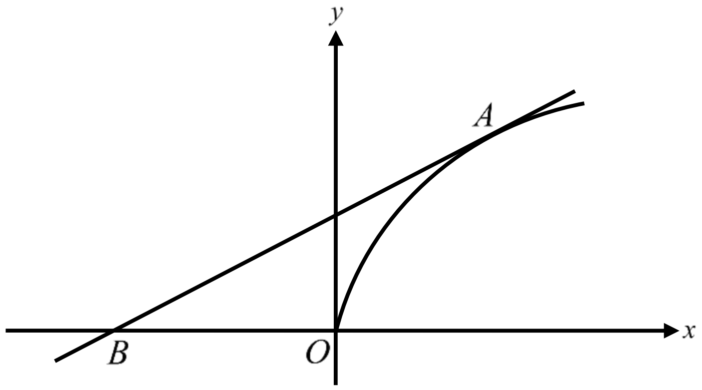
(iii) Find the equation of [PQ] in the form $y = mx + c$.

[8]

(b) Find the area between the graph of f, the tangent at P and the x-axis.

[4]

3. Let $f(x) = x^{\frac{1}{2}}$. The following diagram shows part of the graph of f.



The point $A(h, f(h))$, where $h > 0$, lies on the graph of f. The tangent at A crosses the x-axis at the point $B(b, 0)$.

(a) Express $f'(h)$ in terms of h.

[2]

It is given that the gradient of the tangent at A is $\frac{1}{6}$.

(b) (i) Find h.

(ii) Hence, find the equation of [AB] in the form $y = mx + c$.

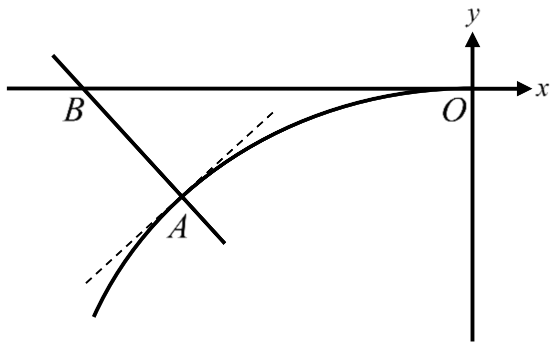
(iii) Find b.

[6]

(c) Find the area between the graph of f, the tangent at A and the x-axis.

[4]

4. Let $f(x) = x^3$. The following diagram shows part of the graph of f.



The point $\mathrm{A}(h, f(h))$, where $h < 0$, lies on the graph of f. The normal at A crosses the x-axis at the point $\mathrm{B}(b, 0)$.

(a) Express $f'(h)$ in terms of h.

[2]

It is given that the gradient of the tangent at A is $\frac{3}{4}$.

(b) (i) Find h.

(ii) Write down the gradient of the normal at A.

(iii) Hence, find the equation of [AB] in the form $y = mx + c$.

(iv) Find b.

[7]

(c) Find the area between the graph of f, the normal at A and the x-axis.

[4]

53 Paper 2 – Problems in Kinematics

Example

A particle moves in a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by

$$v(t)=(t-3)^3\text{, for }0\le t\le 6.$$

(a) Find the velocity of the particle when $t=2$.

[2]

(b) Find the value of t for which the particle is at rest.

[3]

(c) Find the total distance the particle travels during the first six seconds.

[3]

(d) Find the expression of $a(t)$, the acceleration of the particle.

[2]

(e) Find all possible values of t for which the velocity and acceleration are both positive.

[3]

It is given that the initial displacement of the particle is $\frac{81}{4}$ m.

(f) Find the expression of $s(t)$, the displacement of the particle.

[4]

Solution

(a) The initial velocity

$=v(2)$

$=(2-3)^3$ (M1) for substitution

$=-1\text{ ms}^{-1}$ A1

[2]

(b) $v(t)=0$ (M1) for setting equation

$(t-3)^3=0$

$t-3=0$ (A1) for correct approach

$t=3$ A1

[3]

(c) The total distance travelled

$= \int_0^6 |v(t)|\,dt$ (M1) for valid approach

$= \int_0^6 |(t-3)^3|\,dt$ (A1) for substitution

$= 40.5$ m A1

[3]

(d) $a(t) = v'(t)$

$a(t) = 3(t-3)^2(1)$ (A1) for correct approach

$a(t) = 3(t-3)^2$ A1

[2]

(e) $v(t) > 0$ and $a(t) > 0$

By considering the graph of $y = (t-3)^3$ and

$y = 3(t-3)^2$, $t > 3$ and $t \neq 3$. R2

$\therefore 3 < t \leq 6$ A1

[3]

(f) $s(t) = \int v(t)\,dt$

$s(t) = \int (t-3)^3\,dt$ (A1) for correct approach

$s(t) = \frac{1}{4}(t-3)^4 + C$ A1

$\frac{81}{4} = \frac{1}{4}(0-3)^4 + C$ (M1) for substitution

$C = 0$

$\therefore s(t) = \frac{1}{4}(t-3)^4$ A1

[4]

Exercise 53

1. A particle moves in a straight line. Its velocity, v ms^{-1}, at time t seconds, is given by

$$v(t) = -(t-4)^3, \text{ for } 0 \le t \le 8.$$

(a) Find the initial velocity.
[2]

(b) Find the value of t for which the velocity of the particle is -27 ms^{-1}.
[3]

(c) Find the total distance the particle travels during the first seven seconds.
[3]

(d) Find the expression of $a(t)$, the acceleration of the particle.
[2]

(e) Find all possible values of t for which the velocity and acceleration are both negative.
[3]

It is given that the initial displacement of the particle is 28 m.

(f) Find the expression of $s(t)$, the displacement of the particle.
[4]

2. A particle moves in a straight line. Its initial displacement is zero metres. Its velocity, v ms^{-1}, at time t seconds, is given by

$$v(t) = -2t^3 + 12t^2 - 24t + 16, \text{ for } 0 \le t \le 4.$$

(a) Find the expression of $s(t)$, the displacement of the particle.
[4]

(b) Find the exact value of the displacement of the particle at $t = 3.3$.
[2]

(c) Find the total distance the particle travels during the first 3.3 seconds.
[3]

(d) Find the expression of $a(t)$, the acceleration of the particle.
[2]

(e) Find all possible values of t for which the velocity is positive and the acceleration is negative.
[3]

3. A particle moves in a straight line. Its initial displacement is one metre. Its velocity, v ms^{-1}, at time t seconds, is given by

$$v(t) = \pi \cos \pi t, \text{ for } 0 \le t \le 5.$$

(a) (i) Use a suitable substitution, show that $s(t) = \int \cos u \, du$.

(ii) Hence, find the expression of $s(t)$, the displacement of the particle.

[6]

(b) Find the values of t for which the displacement of the particle is 0 m.

[3]

(c) (i) Find the expression of $a(t)$, the acceleration of the particle.

(ii) Hence, find all possible values of t for which its acceleration is positive.

[5]

It is given that there are five stationary points of the graph of $s(t)$, for $0 \le t \le 5$.

(d) Write down the number of times for which the particle is at rest.

[1]

4. A particle moves in a straight line. Its initial displacement is zero metres and its initial velocity is 48 ms^{-1}. Its acceleration, a ms^{-2}, at time t seconds, is given by

$$a(t) = 4t^3 - 33t^2 + 88t - 76, \text{ for } 0 \le t \le 5.$$

(a) Find the expression of $v(t)$, the velocity of the particle.

[4]

(b) Hence, find the expression of $s(t)$, the displacement of the particle.

[4]

(c) Find all possible values of t for which the displacement is positive and the acceleration is negative.

[4]

The table below shows the information of the graph of $v(t)$, for $0 \le t \le 5$.

t	$0 \le t < 2$	$t = 2$	$2 < t < 3$	$t = 3$	$3 < t < 4$	$t = 4$	$4 < t \le 5$
$v(t)$	positive	0	positive	0	negative	0	positive

(d) (i) Write down the number of times for which the particle is at rest.

(ii) Write down the number of times for which the particle changes its direction.

[2]

Chapter

12

Differential Equations

SUMMARY POINTs

✓ $\frac{dy}{dx}=f(x,y)$: First order differential equation

✓ Solving $\frac{dy}{dx}=f(x)g(y)$ by separating variables:

$$\frac{dy}{dx}=f(x)g(y)\Rightarrow\int\frac{1}{g(y)}dy=\int f(x)dx$$

✓ Solving $\frac{dy}{dx}=f(x,y)$ by Euler's method, with (x_0,y_0) and step length h :

$$\begin{cases} x_{n+1}=x_n+h \\ y_{n+1}=y_n+h\left.\frac{dy}{dx}\right|_{(x_n,\,y_n)} \end{cases}$$

SUMMARY POINTs

12

✓ Coupled differential equations:

1. $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ can be expressed as $\begin{pmatrix} \dfrac{dx}{dt} \\ \dfrac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2. λ_1, λ_2: Eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. $\mathbf{v}_1$, $\mathbf{v}_2$: Eigenvectors corresponding to λ_1 and λ_2 respectively

4. $\mathbf{x} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2$: Solution of the system

5. Stable equilibrium if λ_1, $\lambda_2 < 0$ or $\lambda_1 = a + bi$, $\lambda_2 = a - bi$ and $a < 0$

6. Unstable equilibrium if λ_1, $\lambda_2 > 0$ or $\lambda_1 = a + bi$, $\lambda_2 = a - bi$ and $a > 0$

7. Saddle point if $\lambda_1 \lambda_2 < 0$

✓ Solving $\begin{cases} \dfrac{dx}{dt} = ax + by \\ \dfrac{dy}{dt} = cx + dy \end{cases}$ by Euler's method, with (t_0, x_0, y_0) and step length h:

$t_{n+1} = t_n + h$ and $\begin{cases} x_{n+1} = x_n + h \left.\dfrac{dx}{dt}\right|_{(t_n, x_n, y_n)} \\ y_{n+1} = y_n + h \left.\dfrac{dy}{dt}\right|_{(t_n, x_n, y_n)} \end{cases}$

✓ Predator-prey models:

$\begin{cases} \dfrac{dx}{dt} = (a - by)x \\ \dfrac{dy}{dt} = (cx - d)y \end{cases}$, where a, b, c and d are positive constants

✓ The second-order differential equation $\dfrac{d^2x}{dt^2} + a\dfrac{dx}{dt} + bx = 0$ can be expressed as

$\begin{cases} \dfrac{dv}{dt} = -av - bx \\ \dfrac{dx}{dt} = v \end{cases}$

Solutions of Chapter 12

54 Paper 1 – Separating Variables

Example

Consider the differential equation $y(32+x^5)\frac{dy}{dx}=5x^4$, where x, $y\geq 0$.

(a) Express $\int y\,dy$ as an indefinite integral of x.

[2]

(b) Hence, using the substitution $u=32+x^5$ to find the expression of $\frac{1}{2}y^2$, giving the answer in terms of x.

[3]

It is given that $y=0$ when $x=0$.

(c) Find the expression of y.

[3]

Solution

(a) $y(32+x^5)\frac{dy}{dx}=5x^4$

$y\,dy=\frac{5x^4}{(32+x^5)}dx$ (M1) for valid approach

$\therefore \int y\,dy=\int \frac{5x^4}{32+x^5}dx$ A1

[2]

(b) Let $u=32+x^5$.

$\frac{du}{dx}=5x^4 \Rightarrow du=5x^4dx$ A1

$\therefore \int y\,dy=\int \frac{1}{u}du$ (A1) for correct working

$\frac{1}{2}y^2=\ln u+C$

$\frac{1}{2}y^2=\ln(32+x^5)+C$ A1

[3]

(c) $\frac{1}{2}(0)^2 = \ln(32+0^5)+C$ (M1) for substitution

$0 = \ln 32 + C$

$C = -\ln 32$ (A1) for correct value

$\therefore \frac{1}{2}y^2 = \ln(32+x^5) - \ln 32$

$y^2 = 2\ln(32+x^5) - 2\ln 32$

$y = \sqrt{2\ln(32+x^5) - 2\ln 32}$ A1

[3]

Exercise 54

1. Consider the differential equation $\frac{dx}{dt} = \pi^2 x \sin \pi t$, where t, $x > 0$.

(a) Express $\int \frac{1}{x} dx$ as an indefinite integral of t.

[2]

(b) Hence, using the substitution $u = \pi t$ to find the expression of $\ln x$, giving the answer in terms of t.

[3]

It is given that $x = 1$ when $t = 0.5$.

(c) Find the expression of x.

[3]

2. Consider the differential equation $\frac{dy}{dx} = \frac{y}{2x+3}$, where x, $y > 0$.

(a) Express $\int \frac{1}{y} dx$ as an indefinite integral of x.

[2]

(b) Hence, using the substitution $u = 2x+3$ to find the expression of $\ln y$, giving the answer in terms of x.

[3]

It is given that $y = 12$ when $x = 3$.

(c) Find the expression of y.

[3]

3. Consider the differential equation $\frac{dy}{dx}=\frac{2}{y^2}$, where x, $y>0$.

(a) Find the expression of $\frac{1}{3}y^3$, giving the answer in terms of x.

[3]

It is given that $y=9$ when $x=121$.

(b) Find the expression of y.

[3]

(c) Hence, find y when $x=\frac{61}{3}$.

[2]

4. Consider the differential equation $\frac{dy}{dx}=-y^3e^x$, where x, $y>0$.

(a) Find the expression of $\frac{1}{2y^2}$, giving the answer in terms of x.

[3]

It is given that $y=0.25$ when $x=\ln 2$.

(b) Find the expression of y.

[3]

(c) Hence, find y when $x=\ln 12$.

[2]

55 Paper 1 – Coupled Differential Equations

Example

Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2x - 15y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -x + 4y \end{cases}$. It can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices.

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find $\det(\mathbf{M} - \lambda \mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(d) Hence, write down the general solution for $\mathbf{X}$.

[2]

Solution

(a) $\det(\mathbf{M} - \lambda \mathbf{I}) = \begin{vmatrix} 2-\lambda & -15 \\ -1 & 4-\lambda \end{vmatrix}$ (M1) for valid approach

$\det(\mathbf{M} - \lambda \mathbf{I}) = (2-\lambda)(4-\lambda) - (-15)(-1)$

$\det(\mathbf{M} - \lambda \mathbf{I}) = 8 - 2\lambda - 4\lambda + \lambda^2 - 15$

$\det(\mathbf{M} - \lambda \mathbf{I}) = \lambda^2 - 6\lambda - 7$ A1

[2]

(b) $\lambda_1 = -1$, $\lambda_2 = 7$ A2

[2]

(c) $\mathbf{v}_1 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ A2

[2]

(d) $\mathbf{X} = Ae^{-t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + Be^{7t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ A2

[2]

Exercise 55

1. Consider the coupled differential equations $\begin{cases} \frac{dx}{dt} = 5x + 6y \\ \frac{dy}{dt} = 4x + 3y \end{cases}$. It can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices.

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(d) Hence, write down the general solution for $\mathbf{X}$.

[2]

2. Consider the coupled differential equations $\begin{cases} \frac{dx}{dt} = 2x + 15y \\ \frac{dy}{dt} = 2x + y \end{cases}$. It can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices.

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(a) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(d) Hence, write down the general solution for

(i) x;

(ii) y.

[2]

3. Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 4x + 2y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -9x - 2y \end{cases}$. It can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{M}\mathbf{X}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices.

(a) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of the eigenvalues of $\mathbf{M}$.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$.

(c) Write down the general solution for $\mathbf{X}$.

[2]

The phase portrait for the coupled differential equation is in the shape of spiral.

(d) Describe the stability of the phase portrait.

[1]

4. Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -8x - y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 13x - 2y \end{cases}$. It can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices.

(a) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(b) Hence, write down the values of the eigenvalues of $\mathbf{M}$.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$.

(c) Write down the general solution for $\mathbf{X}$.

[2]

The phase portrait for the coupled differential equation is in the shape of spiral.

(d) Describe the stability of the phase portrait.

[1]

56 Paper 1 – Euler's Method

Example

Consider the differential equation $\frac{dy}{dx} = \frac{y}{4x}$, where $y = 3$ when $x = 2$. Euler's method with a step length of 0.1 is used to approximate the value of y when $x = 2.3$.

(a) Find the approximate value of y when $x = 2.1$.

[3]

(b) Hence, write down the approximate value of y when

(i) $x = 2.2$, giving the answer correct to 5 decimal places;

(ii) $x = 2.3$.

[2]

Solution

(a) $\begin{cases} x_{n+1} = x_n + 0.1 \\ y_{n+1} = y_n + 0.1 \left.\frac{dy}{dx}\right|_{(x_n,\, y_n)} \end{cases}$ (M1) for valid approach

$x_0 = 2,\ y_0 = 3$ (A1) for correct values

$x_1 = 2 + 0.1 = 2.1$

$y_1 = 3 + 0.1\left(\frac{3}{4(2)}\right) = 3.0375$ A1

[3]

(b) (i) $y_2 = 3.073660714$

$y_2 = 3.07366$ A1

(ii) $y_3 = 3.108588677$

$y_3 = 3.11$ A1

[2]

Exercise 56

1. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{xy}{x^2 - 9}$, where $y = 10$ when $x = 4$. Euler's method with a step length of 0.1 is used to approximate the value of y when $x = 4.3$.

(a) Find the approximate value of y when $x = 4.1$.

[3]

(b) Hence, write down the approximate value of y when

(i) $x = 4.2$, giving the answer correct to 5 decimal places;

(ii) $x = 4.3$.

[2]

2. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = e^x + 4y$, where $y = 2$ when $x = 0$. Euler's method with a step length of 0.2 is used to approximate the value of y when $x = 0.8$.

(a) Find the approximate value of y when $x = 0.2$.

[3]

(b) Hence, write down the approximate value of y when

(i) $x = 0.4$, giving the answer correct to 6 decimal places;

(ii) $x = 0.6$, giving the answer correct to 6 decimal places;

(iii) $x = 0.8$.

[3]

3. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = \frac{1}{x^3}$, where $y = 2$ when $x = 1$. Euler's method with a step length of 0.25 is used to approximate the value of y when $x = 2$.

(a) Find the approximate value of y when $x = 1.25$.

[3]

(b) Hence, write down the approximate value of y when

(i) $x = 1.5$;

(ii) $x = 1.75$, giving the answer correct to 4 decimal places;

(iii) $x = 2$.

[3]

4. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + x - xy = 0$, where $y = 3$ when $x = 3$. Euler's method with a step length of 0.2 is used to approximate the value of y when $x = 4$.

(a) Find the approximate value of y when $x = 3.2$.

[3]

(b) Hence, write down the approximate value of y when

(i) $x = 3.4$;

(ii) $x = 3.6$, giving the answer correct to 4 decimal places;

(ii) $x = 3.8$, giving the answer correct to 4 decimal places;

(iii) $x = 4$.

[4]

57 Paper 1 – More about Euler's Method

Example

Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (x+y)\sin t \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (x+y)\cos t \end{cases}$. $x = 3$ and $y = 4$ when $t = 0$. Euler's method with a step length of 0.1 is used to approximate the values of x and y when $t = 0.2$.

(a) Find, when $t = 0.1$, the approximate value of

(i) x;

(ii) y.

[4]

(b) Hence, write down, when $t = 0.2$, the approximate value of

(i) x;

(ii) y.

[2]

Solution

(a) (i)
$$\begin{cases} x_{n+1} = x_n + 0.1\left.\dfrac{\mathrm{d}x}{\mathrm{d}t}\right|_{(t_n,\, x_n,\, y_n)} \\ y_{n+1} = y_n + 0.1\left.\dfrac{\mathrm{d}y}{\mathrm{d}t}\right|_{(t_n,\, x_n,\, y_n)} \\ t_{n+1} = t_n + 0.1 \end{cases}$$
(M1) for valid approach

$t_0 = 0$, $x_0 = 3$, $y_0 = 4$ (A1) for correct values

$t_1 = 0 + 0.1 = 0.1$

$x_1 = 3 + 0.1\big((3+4)\sin 0\big) = 3$ A1

(ii) $y_1 = 4 + 0.1\big((3+4)\cos 0\big) = 4.7$ A1

[4]

(b) (i) $x_2 = 3.076871731$

$x_2 = 3.08$ A1

(ii) $y_2 = 5.466153207$

$y_2 = 5.47$ A1

[2]

Exercise 57

1. Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x + yt \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = xt + y \end{cases}$. $x = 5$ and $y = 2$ when $t = 1$.

Euler's method with a step length of 0.15 is used to approximate the values of x and y when $t = 1.3$.

(a) Find, when $t = 1.15$, the approximate value of

(i) x;

(ii) y.

[4]

(b) Hence, write down, when $t = 1.3$, the approximate value of

(i) x;

(ii) y.

[2]

2. Consider the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (2 - y)t \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (2 - x)t \end{cases}$. $x = 1.1$ and $y = 1$ when $t = 0.97$. Euler's method with a step length of 0.01 is used to approximate the values of x and y when $t = 1$.

(a) Find, when $t = 0.98$, the approximate value of

(i) x;

(ii) y.

[4]

(b) Hence, write down, the approximate value of

(i) x when $t = 0.99$;

(ii) y when $t = 0.99$;

(iii) x when $t = 1$;

(ii) y when $t = 1$.

[4]

3. Consider the differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 15x = 0$. $x = 0$ and $\frac{dx}{dt} = 1$ when $t = 2$. Euler's method with a step length of 0.1 is used to approximate the value of x when $t = 2.2$.

(a) By using $u = \frac{dx}{dt}$, express the differential equation in a coupled system.

[1]

(b) Find, when $t = 2.1$, the approximate value of

(i) u;

(ii) x.

[4]

(c) Hence, write down, when $t = 2.2$, the approximate value of x.

[1]

4. Consider the differential equation $\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 36x = 0$. $x = 2$ and $\frac{dx}{dt} = 3$ when $t = 0$. Euler's method with a step length of 0.05 is used to approximate the value of x when $t = 0.1$.

(a) By using $u = \frac{dx}{dt}$, express the differential equation in a coupled system.

[1]

(b) Find, when $t = 0.05$, the approximate value of

(i) u;

(ii) x.

[4]

(c) Hence, write down, when $t = 0.1$, the approximate value of x.

[1]

58 Paper 1 – Predator-Prey Models

Example

Let x and y be the populations, in thousands, of tiger and lion in a forest respectively. The changes in the populations can be modelled by the coupled differential equations

$$\begin{cases} \dfrac{dx}{dt} = (4-y)x \\ \dfrac{dy}{dt} = (2x-5)y \end{cases}.$$

(a) State the equilibrium points for the two populations.

[2]

The initial populations of tiger and lion are 1200 and 1000 respectively.

(b) (i) Express $\dfrac{dy}{dx}$ in terms of x and y.

(ii) Hence, find the value of $\dfrac{dy}{dx}$ when $t=0$.

(iii) State, at the beginning, whether the population of tiger is increasing or decreasing.

[4]

Solution

(a)		$(0, 0)$, $(2.5, 4)$	A2
			[2]
(b)	(i)	$\dfrac{dy}{dx} = \dfrac{(2x-5)y}{(4-y)x}$	A1
	(ii)	$\left.\dfrac{dy}{dx}\right\rvert_{t=0} = \dfrac{(2(1.2)-5)(1)}{(4-1)(1.2)}$	(M1) for substitution
		$\left.\dfrac{dy}{dx}\right\rvert_{t=0} = -\dfrac{13}{18}$	A1
	(iii)	The population of tiger is increasing at the beginning.	A1
			[4]

Exercise 58

1. Let x and y be the populations, in thousands, of shark and whale in an ocean respectively. The changes in the populations can be modelled by the coupled differential equations $\begin{cases} \frac{dx}{dt} = (3-2y)x \\ \frac{dy}{dt} = (x-6)y \end{cases}$.

(a) State the equilibrium points for the two populations.

[2]

The initial populations of shark and whale are 2000 and 2500 respectively.

(b) (i) Express $\frac{dy}{dx}$ in terms of x and y.

(ii) Hence, find the value of $\frac{dy}{dx}$ when $t = 0$.

(iii) State, at the beginning, whether the population of shark is increasing or decreasing.

[4]

2. Let x and y be the number of soldiers, in thousands, of army A and army B respectively. The changes in the number of soldiers in a battle between them can be modelled by the coupled differential equations $\begin{cases} \frac{dx}{dt} = (2-5y)x \\ \frac{dy}{dt} = (x-8)y \end{cases}$.

(a) State the equilibrium points for the two populations.

[2]

The initial number of soldiers of army A and army B are 4000 and 3000 respectively.

(b) (i) Express $\frac{dy}{dx}$ in terms of x and y.

(ii) Hence, find the value of $\frac{dy}{dx}$ when $t = 0$.

(iii) When $x > 0$, state the range of values of y such that $\frac{dx}{dt} > 0$.

[4]

3. Let x and y be the populations, in thousands, of leopard and crocodile in a habitat respectively. The changes in the populations can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (5-8y)x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (7x-3)y \end{cases}$.

The initial populations of leopard and crocodile are both 500.

(a) State, at the beginning, whether the population of crocodile is increasing or decreasing.

[1]

Euler's method with a step length of 0.2 is used to approximate the populations of leopard and crocodile when $t = 0.2$.

(b) Find, when $t = 0.2$, the approximate value of the population of

(i) leopard;

(ii) crocodile.

[4]

4. Let x and y be the number of soldiers, in hundreds, from country X and country Y respectively. The changes in the number of soldiers in a battle between the two countries can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (12-5y)x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = (3x-10)y \end{cases}$.

(a) When $y > 0$, state the range of values of x such that $\dfrac{\mathrm{d}y}{\mathrm{d}t} > 0$.

[1]

The initial numbers of soldiers from the two countries are both 300. Euler's method with a step length of 0.1 is used to approximate the numbers of soldiers from the two countries when $t = 0.1$.

(b) Find, when $t = 0.1$, the approximate numbers of soldiers from

(i) country X;

(ii) country Y.

[4]

59 Paper 2 – Miscellaneous Problems

Example

Under the influence of a magnetic field, the changes in the position (x, y) of a metal ball can be modelled by the coupled differential equations $\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = 3x + 2y \\ \frac{\mathrm{d}y}{\mathrm{d}t} = 4x + y \end{cases}$.

(a) Write down the coordinates of the equilibrium point.

[1]

The system can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(b) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(c) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

The metal ball is initially at $(2, 5)$.

(e) Find the particular solution of

(i) x;

(ii) y.

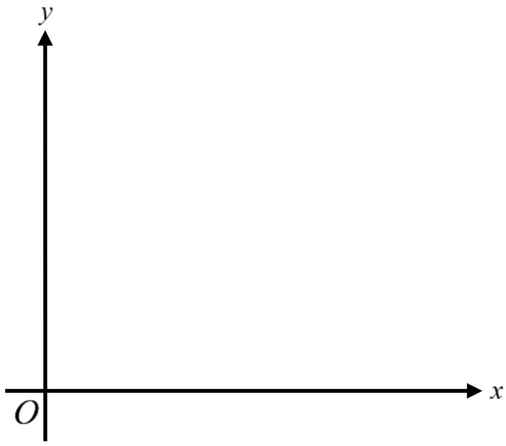
[5]

(f) Hence, find the coordinates of the position of the metal ball at $t = 0.5$.

[2]

(g) On the following diagram, sketch the following trajectory for this particular solution.

[2]



Solution

(a) (0, 0) A1

[1]

(b) $\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 2\\ 4 & 1-\lambda\end{vmatrix}$ (M1) for valid approach

$\det(\mathbf{M}-\lambda\mathbf{I})=(3-\lambda)(1-\lambda)-(2)(4)$

$\det(\mathbf{M}-\lambda\mathbf{I})=3-3\lambda-\lambda+\lambda^2-8$

$\det(\mathbf{M}-\lambda\mathbf{I})=\lambda^2-4\lambda-5$ A1

[2]

(c) $\lambda_1=-1,\ \lambda_2=5$ A2

[2]

(d) $\mathbf{v}_1=\begin{pmatrix}-\frac{1}{2}\\ 1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\ 1\end{pmatrix}$ A2

[2]

(e) (i) $\mathbf{X} = Ae^{-t}\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + Be^{5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (A1) for correct approach

$\begin{pmatrix} 2 \\ 5 \end{pmatrix} = Ae^{-0}\begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix} + Be^{5(0)}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (M1) for substitution

$$\begin{cases} 2 = -\frac{1}{2}A + B \\ 5 = A + B \end{cases}$$

By solving this system, $A = 2$ and $B = 3$. (A1) for correct values

$\therefore x = -e^{-t} + 3e^{5t}$ A1

(ii) $y = 2e^{-t} + 3e^{5t}$ A1

[5]

(f) The required coordinates

$= (-e^{-0.5} + 3e^{5(0.5)}, 2e^{-0.5} + 3e^{5(0.5)})$ (M1) for substitution

$= (35.94095122, 37.7605432)$

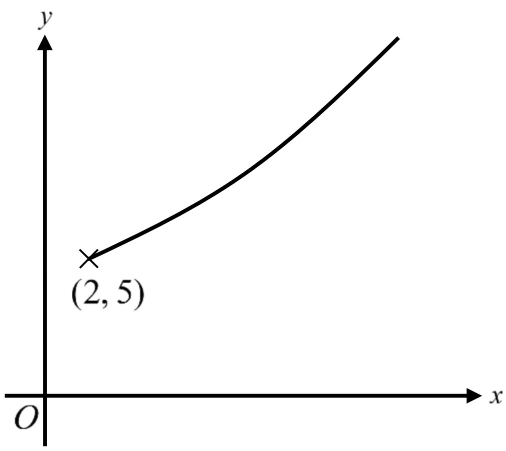
$= (35.9, 37.8)$ A1

[2]

(g) For starting at $(2, 5)$ A1

For positive gradient A1

[2]



Exercise 59

1. In a physics experiment, the changes in the position (x, y) of a particle can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -x - 5y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x - 7y \end{cases}$.

(a) Write down the coordinates of the equilibrium point.

[1]

The system can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(b) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(c) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

The particle is initially at $(9, 1)$.

(e) Find the particular solution of

(i) x;

(ii) y.

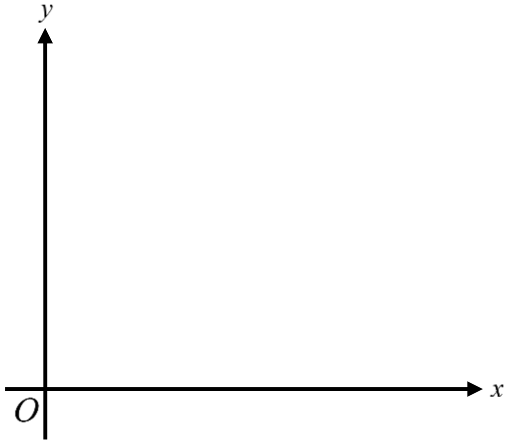
[5]

(f) Hence, find the coordinates of the position of the particle at $t = 1$.

[2]

(g) On the following diagram, sketch the following trajectory for this particular solution.

[2]



2. Let x and y be the populations, in thousands, of horses and zebras in a national park respectively. The changes in the populations can be modelled by the coupled differential equations $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = -3x \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = 4y - x \end{cases}$.

(a) When $x = 2$, state the range of values of y such that $\dfrac{\mathrm{d}y}{\mathrm{d}t} > 10$.

[1]

The system can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \dfrac{\mathrm{d}x}{\mathrm{d}t} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(b) Find $\det(\mathbf{M} - \lambda \mathbf{I})$, giving the answer in terms of λ.

[2]

(c) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

The initial populations of horse and zebra are 7000 and 2000 respectively.

(e) Find the particular solution of

(i) x;

(ii) y.

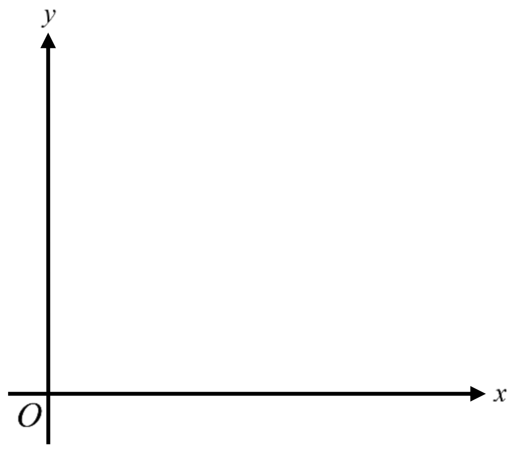
[5]

(f) Hence, state the long-term behaviour for the population of horse.

[1]

(g) On the following diagram, sketch the following trajectory for this particular solution.

[2]



3. A particle moves in a straight line with velocity v ms^{-1} and displacement x m with respect to the starting point O. By considering the rate of change of its velocity, the relationship between the variables can be modelled by the differential equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} - 4x = 0.$$

(a) By using $v = \frac{dx}{dt}$, express the differential equation in a coupled system.

[1]

Euler's method with a step length of 0.1 is used to approximate the displacement of the particle at $t = 0.3$. It is given that initially the particle is at O with velocity 1 ms^{-1}.

(b) Find, when $t = 0.1$, the approximate value of

(i) v;

(ii) x.

[4]

(c) Hence, write down the approximate value of the displacement at

(i) $t = 0.2$;

(ii) $t = 0.3$.

[2]

The system can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} v \\ x \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(d) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(e) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(f) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(g) Find

(i) the particular solution of x;

(ii) the displacement at $t = 0.3$.

[6]

(h) Hence, write down the difference between the approximated displacement in (c)(ii) and the actual displacement in (g)(ii).

[1]

4. A box connected with a spring moves with velocity v ms^{-1} and displacement x m with respect to the starting point O. By considering the rate of change of its velocity, the relationship between the variables can be modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 9x = 0.$$

(a) By using $v = \frac{\mathrm{d}x}{\mathrm{d}t}$, express the differential equation in a coupled system.

[1]

Euler's method with a step length of 0.25 is used to approximate the displacement of the particle at $t = 1$. It is given that initially the particle is at rest with displacement 1 m.

(b) Find, when $t = 0.25$, the approximate value of

(i) v;

(ii) x.

[4]

(c) Hence, write down the approximate value of the displacement at

(i) $t = 0.5$;

(ii) $t = 0.75$;

(iii) $t = 1$.

[3]

The system can be expressed by a matrix equation $\dot{\mathbf{X}} = \mathbf{MX}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{X}} = \begin{pmatrix} \frac{\mathrm{d}v}{\mathrm{d}t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} v \\ x \end{pmatrix}$ are two 2×1 matrices. Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(d) Find $\det(\mathbf{M} - \lambda\mathbf{I})$, giving the answer in terms of λ.

[2]

(e) Hence, write down the values of λ_1 and λ_2.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(f) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

(g) Find

(i) the particular solution of x;

(ii) the displacement at $t = 1$.

[6]

(h) Hence, calculate the percentage error for the approximated displacement in (c)(ii).

[2]

Chapter

13

Probability

SUMMARY POINTs

✓ Markov Chain with transition matrix $\mathbf{T}$:

1. $\det(\mathbf{T}-\lambda\mathbf{I})$: Characteristic polynomial of $\mathbf{T}$
2. Solution(s) of $\det(\mathbf{T}-\lambda\mathbf{I})=0$: Eigenvalue(s) of $\mathbf{T}$
3. $\mathbf{v}$: Steady state probability vector, which is the eigenvector of $\mathbf{T}$ corresponding to the eigenvalue $\lambda=1$
4. $\mathbf{v}_0$: Initial state probability vector
5. $\mathbf{v}_n=\mathbf{T}^n\mathbf{v}_0$: State probability vector after n transitions
6. The column sum of $\mathbf{T}$ must be equal to 1

Solutions of Chapter 13

60 Paper 1 – Markov Chain

Example

The matrix $\mathbf{T}=\begin{pmatrix}0.2 & 0.75\\ 0.8 & 0.25\end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

Solution

(a) The characteristic polynomial of $\mathbf{T}$

$=\det(\mathbf{T}-\lambda\mathbf{I})$

$=\begin{vmatrix}0.2-\lambda & 0.75\\ 0.8 & 0.25-\lambda\end{vmatrix}$ (M1) for valid approach

$=(0.2-\lambda)(0.25-\lambda)-(0.75)(0.8)$

$=0.05-0.2\lambda-0.25\lambda+\lambda^2-0.6$

$=\lambda^2-0.45\lambda-0.55$ A1

[2]

(b) $\lambda_1=-\dfrac{11}{20}$, $\lambda_2=1$ A2

[2]

(c) $\mathbf{v}$ is the eigenvector of $\mathbf{T}$ corresponding to $\lambda_2=1$. (R1) for correct reasoning

$\therefore \mathbf{v}=\begin{pmatrix}\dfrac{15}{31}\\ \dfrac{16}{31}\end{pmatrix}$ A1

[2]

Exercise 60

1. The matrix $\mathbf{T}=\begin{pmatrix}0.05 & 0.3\\0.95 & 0.7\end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1<\lambda_2$.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

2. The matrix $\mathbf{T}=\begin{pmatrix}0.5 & 0.15\\0.5 & 0.85\end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(b) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1<\lambda_2$.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

3. The matrix $\mathbf{T}=\begin{pmatrix}0.17 & 0.9\\0.83 & 0.1\end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1<\lambda_2$.

[3]

Let $\mathbf{v}_0=\begin{pmatrix}0.5\\0.5\end{pmatrix}$.

(b) Find $\mathbf{v}_8$, the state probability vector after 8 transitions.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

4. The matrix $\mathbf{T} = \begin{pmatrix} 0.52 & 0.63 \\ 0.48 & 0.37 \end{pmatrix}$ is a transition matrix for a Markov chain.

(a) Find the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[3]

Let $\mathbf{v}_0 = \begin{pmatrix} 0.09 \\ 0.91 \end{pmatrix}$.

(b) Find $\mathbf{v}_{13}$, the state probability vector after 13 transitions.

[2]

(c) Find $\mathbf{v}$, the steady state probability vector for this Markov chain.

[2]

61 Paper 2 – Real Life Problems

Example

X and Y are the only two cities on an island. Each month 30% of the residents in X move to Y, and 55% of the residents in Y choose to stay in Y. Assume that there is no net gain or net loss of the entire population on the island.

(a) Find $\mathbf{T}$, the transition matrix representing the changes in population between X and Y in a particular month.

[2]

(b) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(c) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{T}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{T}^n$ in terms of n.

[3]

The initial population of X and Y are 550 and 450 respectively.

(g) Find, after a year, the population of X and Y.

[4]

Solution

(a) $\mathbf{T}=\begin{pmatrix}0.7 & 0.45\\ 0.3 & 0.55\end{pmatrix}$ A2

[2]

(b) The characteristic polynomial of $\mathbf{T}$

$=\det(\mathbf{T}-\lambda\mathbf{I})$

$=\begin{vmatrix}0.7-\lambda & 0.45\\ 0.3 & 0.55-\lambda\end{vmatrix}$ (M1) for valid approach

$=(0.7-\lambda)(0.55-\lambda)-(0.45)(0.3)$

$=0.385-0.7\lambda-0.55\lambda+\lambda^2-0.135$

$=\lambda^2-1.25\lambda+0.25$ A1

[2]

(c) $\lambda_1=\frac{1}{4},\ \lambda_2=1$ A2

[2]

(d) $\mathbf{v}_1=\begin{pmatrix}1\\ -1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\ \frac{2}{3}\end{pmatrix}$ A2

[2]

(e) (i) $\begin{pmatrix}1 & 1\\ -1 & \frac{2}{3}\end{pmatrix}$ A1

(ii) $\begin{pmatrix}\left(\frac{1}{4}\right)^n & 0\\ 0 & 1\end{pmatrix}$ A2

[3]

(f) $\mathbf{T}^n=\begin{pmatrix}1 & 1\\ -1 & \frac{2}{3}\end{pmatrix}\begin{pmatrix}\left(\frac{1}{4}\right)^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & \frac{2}{3}\end{pmatrix}^{-1}$

$\mathbf{T}^n=\begin{pmatrix}\left(\frac{1}{4}\right)^n & 1\\ -\left(\frac{1}{4}\right)^n & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & \frac{2}{3}\end{pmatrix}^{-1}$ A1

$\mathbf{T}^n=\begin{pmatrix}\left(\frac{1}{4}\right)^n & 1\\ -\left(\frac{1}{4}\right)^n & \frac{2}{3}\end{pmatrix}\begin{pmatrix}\frac{2}{5} & -\frac{3}{5}\\ \frac{3}{5} & \frac{3}{5}\end{pmatrix}$ (A1) for correct approach

$$\mathbf{T}^n = \begin{pmatrix} \frac{2}{5}\left(\frac{1}{4}\right)^n + \frac{3}{5} & -\frac{3}{5}\left(\frac{1}{4}\right)^n + \frac{3}{5} \\ -\frac{2}{5}\left(\frac{1}{4}\right)^n + \frac{2}{5} & \frac{3}{5}\left(\frac{1}{4}\right)^n + \frac{2}{5} \end{pmatrix}$$ A1

[3]

(g) $$\mathbf{T}^{12}\begin{pmatrix} 550 \\ 450 \end{pmatrix} = \begin{pmatrix} \frac{2}{5}\left(\frac{1}{4}\right)^{12} + \frac{3}{5} & -\frac{3}{5}\left(\frac{1}{4}\right)^{12} + \frac{3}{5} \\ -\frac{2}{5}\left(\frac{1}{4}\right)^{12} + \frac{2}{5} & \frac{3}{5}\left(\frac{1}{4}\right)^{12} + \frac{2}{5} \end{pmatrix}\begin{pmatrix} 550 \\ 450 \end{pmatrix}$$ M1A1

$$\mathbf{T}^{12}\begin{pmatrix} 550 \\ 450 \end{pmatrix} = \begin{pmatrix} 599.999997 \\ 400.000003 \end{pmatrix}$$

Thus, the population of X and Y after a year are 600 and 400 respectively. A2

[4]

Exercise 61

1. In a town, citizens purchase computers from either Japanese manufacturers or American manufacturers. Each year 25% of the citizens purchasing computers from American manufacturers change their choices to Japanese manufacturers, and 70% of the citizens purchasing computers from Japanese manufacturers do not change their choices. Assume that there is no net gain or net loss of the entire population in the town.

(a) Find $\mathbf{T}$, the transition matrix representing the changes in the number of citizens choosing manufacturers in a particular year.

[2]

(b) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(c) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{T}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{T}^n$ in terms of n.

[3]

The initial number of citizens choosing each manufacturers from the two countries are both 3500.

(g) Find, after five years, the number of citizens purchasing computers from American manufacturers.

[4]

2. The weather condition of a town is studied every day. Assume that the weather is either sunny or cloudy. If it is sunny on a particular day, then there will be 40% of chance that the next day is cloudy. If it is cloudy on a particular day, then there will be 36% of chance that the next day is sunny.

(a) Find $\mathbf{T}$, the transition matrix representing the changes in the weather conditions on a particular day.

[2]

(b) Find the characteristic polynomial of $\mathbf{T}$.

[2]

(c) Hence, write down the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[2]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{T}$ corresponding to λ_1 and λ_2 respectively.

(d) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(e) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(f) Hence, express $\mathbf{T}^n$ in terms of n.

[3]

On a particular day, the probability that it is sunny is two times the probability that it is cloudy.

(g) Find, after a week, the probability that it is cloudy.

[4]

3. A district is served by two supermarkets, A and B. Customers living in the district can choose buying in either A or B. Each week 82% of the customers choosing A and 87% of the customers choosing B will not change their choices in the next week. Assume that there is no net gain or net loss of the total number of customers.

(a) Find $\mathbf{T}$, the transition matrix representing the changes in the number of customers choosing supermarkets in a particular week.

[2]

(b) Find the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[3]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{T}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{T}^n$ in terms of n.

[3]

The initial number of customers choosing each supermarket are both 310.

(f) Write down $\mathbf{T}^n\begin{pmatrix}310\\310\end{pmatrix}$.

[2]

(g) Hence, write down, after a long term, the numbers of customers choosing

(i) A;

(ii) B.

[2]

4. The proportion of a group of people supporting the political parties, A and B, are studied. Each year 10% of the people supporting A and 7% of the people supporting B will change their preferences in the next year. It is assumed that people can either choose supporting A or B, and there is no change in the number of people in the group.

(a) Find $\mathbf{T}$, the transition matrix representing the changes in the proportion of political party supporters in a particular year.

[2]

(b) Find the values of λ_1 and λ_2, the eigenvalues of $\mathbf{T}$, where $\lambda_1 < \lambda_2$.

[3]

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{T}$ corresponding to λ_1 and λ_2 respectively.

(c) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

[2]

It is given that $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$, where $\mathbf{P}$ is a 2×2 matrix and $\mathbf{D}$ is a 2×2 diagonal matrix.

(d) Write down

(i) $\mathbf{P}$;

(ii) $\mathbf{D}^n$.

[3]

(e) Hence, express $\mathbf{T}^n$ in terms of n.

[3]

It is given that the ratio of the number of supporters of A to that of B is 3:2 in 2019.

(f) Write down $\mathbf{T}^n \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix}$.

[2]

(g) Hence, find the proportion of the supporters of A in 2013.

[2]

(h) Using (f) to write down, after a long term, the proportions of supporters of

(i) A;

(ii) B.

[2]

Chapter 14

Poisson Distribution

SUMMARY POINTs

✓ Properties of a random variable $X \sim \text{Po}(\lambda)$ following Poisson distribution:

1. The expected number of occurrences of an event is directly proportional to the length of the time interval
2. The numbers of occurrences of the event in different disjoint time intervals are independent
3. λ: Mean number of occurrences of an event
4. X: Number of occurrences of an event

✓ Formulae for Poisson distribution:

1. $\text{P}(X = r) = \frac{e^{-\lambda} \cdot \lambda^r}{r!}$ for $r \ge 0$, $r \in \mathbb{Z}$
2. $\text{E}(X) = \lambda$: Expected value of X
3. $\text{Var}(X) = \lambda$: Variance of X
4. $\sqrt{\lambda}$: Standard deviation of X
5. $\text{P}(X \le r) = \text{P}(X < r+1) = 1 - \text{P}(X \ge r+1)$

Solutions of Chapter 14

62 Paper 1 – Evaluating Probabilities

Example

The weekly number of traffic accidents occurred in a certain road follows a Poisson distribution with mean 0.45.

(a) Find the probability that at least two traffic accidents will occur in this road in a particular week.

[2]

(b) Find the probability that exactly five traffic accidents will occur in this road in a particular four-week period.

[2]

(c) Given that in a particular four-week period at most five traffic accidents occur, find the probability that less than four traffic accidents occur.

[3]

Solution

(a) $X \sim \text{Po}(0.45)$

$\text{P}(X \geq 2) = 1 - \text{P}(X \leq 1)$ (M1) for valid approach

$\text{P}(X \geq 2) = 1 - 0.9245608199$

$\text{P}(X \geq 2) = 0.0754391801$

$\text{P}(X \geq 2) = 0.0754$ A1

[2]

(b) $Y \sim \text{Po}(1.8)$ (M1) for valid approach

$\text{P}(Y = 5) = 0.0260286241$

$\text{P}(Y = 5) = 0.0260$ A1

[2]

(c) $\text{P}(Y < 4 \mid Y \leq 5) = \dfrac{\text{P}(Y < 4 \cap Y \leq 5)}{\text{P}(Y \leq 5)}$ (A1) for substitution

$\text{P}(Y < 4 \mid Y \leq 5) = \dfrac{\text{P}(Y \leq 3)}{\text{P}(Y \leq 5)}$

$\text{P}(Y < 4 \mid Y \leq 5) = \dfrac{0.8912916054}{0.9896219631}$ (A1) for correct approach

$\text{P}(Y < 4 \mid Y \leq 5) = 0.9006384646$

$\text{P}(Y < 4 \mid Y \leq 5) = 0.901$ A1

[3]

Exercise 62

1. The number of goals scored in a randomly selected match by a football team follows a Poisson distribution with mean 1.7.

(a) Find the probability that less than two goals are scored in a particular match.

[2]

(b) Find the probability that exactly ten goals are scored in six randomly selected matches.

[2]

(c) Given that in six randomly selected matches less than nine goals are scored, find the probability that less than six goals are scored.

[3]

2. In a hotline centre, the number of incoming calls is distributed as Poisson with mean 12.8 in a 30-minute time period.

(a) Find the probability that more than 10 incoming calls are received in a particular 30-minute time period.

[2]

(b) Find the probability that at most 33 incoming calls are received in a particular 90-minute time period.

[2]

(c) Given that a particular 90-minute time period at least 40 incoming calls are received, find the probability that exactly 43 incoming calls are received.

[3]

3. The number of delays in a day of a railway system follows a Poisson distribution with mean 3.1.

(a) Find the probability that there are at least 2 delays in a day.

[2]

(b) Find the probability that there are at most 28 delays in a week.

[2]

(c) Find the probability that in 7 consecutive days there are at most 4 delays in each day.

[2]

4. The number of accidents occurs in the park X follows a Poisson distribution with mean 7 accidents per year.

(a) Find the probability that there are more than 7 accidents in the park X in a particular year.

[2]

(b) Find the probability that there are not more than 2 accidents in the park X in a particular 3-month period.

[2]

The number of accidents occurs in the other park Y follows the Poisson distribution with mean 0.5 accidents per month.

(c) Find the probability that there are 5 accidents in each park in a particular year.

[2]

63 Paper 1 – Mean and Variance

Example

The random variable X follows a Poisson distribution with mean a. It is given that $24.5\mathrm{P}(X=0)=\mathrm{P}(X=2)$.

(a) Find a.

[3]

(b) Hence, write down the values of

(i) $\mathrm{P}(X=3)$;

(ii) the variance of X.

[2]

Solution

(a) $24.5\mathrm{P}(X=0)=\mathrm{P}(X=2)$

$\therefore 24.5\left(\frac{e^{-a}\cdot a^0}{0!}\right)=\frac{e^{-a}\cdot a^2}{2!}$ (M1) for setting equation

$24.5e^{-a}=\frac{a^2e^{-a}}{2}$ (A1) for correct approach

$a^2=49$

$a=-7$ (*Rejected*) or $a=7$ A1

[3]

(b) (i) $\mathrm{P}(X=3)=0.0521292524$

$\mathrm{P}(X=3)=0.0521$ A1

(ii) 7 A1

[2]

Exercise 63

1. The random variable X follows a Poisson distribution with mean b. It is given that $\mathrm{P}(X=0)=\frac{3}{32}\mathrm{P}(X=3)$.

(a) Find b.

[3]

(b) Hence, write down the values of

(i) $\mathrm{P}(X=2)$;

(ii) the variance of X.

[2]

2. The random variable X follows a Poisson distribution with mean c. It is given that $9\mathrm{P}(X=1)=2\mathrm{P}(X=2)$.

(a) Find c.

[3]

(b) Hence, write down the values of

(i) $\mathrm{P}(X\le 4)$;

(ii) the standard deviation of X.

[2]

3. The number of digital cameras sold in a store each week follows a Poisson distribution with mean λ, where $\lambda>12$, $\lambda\in\mathbb{Z}$. The probability that 13 digital cameras are sold in a particular week is 0.0956.

(a) Find λ.

[3]

(b) Hence, find the probability that more than 3 digital cameras are sold on a particular day.

[2]

4. The number of typing errors made by Mr. Lee in the book Your Practice Set – Applications and Interpretation Book 2 follows a Poisson distribution with mean λ per page, where $0<\lambda<1$, $\lambda\in\mathbb{R}$. The probability that exactly 1 typing error is made on a particular page is 0.00990.

(a) Find λ.

[3]

(b) Hence, find the probability that more than 2 typing errors are made on a particular 120-page section.

[2]

Chapter

15

Linear Combinations of Variables

15

SUMMARY POINTs

✓ Properties of linear transformations of independent random variables:

1. $\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$
2. $\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$
3. $\mathrm{E}(a_1X_1+a_2X_2+\cdots+a_nX_n)=a_1\mathrm{E}(X_1)+a_2\mathrm{E}(X_2)+\cdots+a_n\mathrm{E}(X_n)$
4. $\mathrm{Var}(a_1X_1+a_2X_2+\cdots+a_nX_n)=a_1^2\mathrm{Var}(X_1)+a_2^2\mathrm{Var}(X_2)+\cdots+a_n^2\mathrm{Var}(X_n)$

Solutions of Chapter 15

64 Paper 1 – Expected Value and Variance

Example

The random variable X is defined such that $\mathrm{E}(X)=1.6$ and $\mathrm{Var}(X)=0.25$.

(a) Find $\mathrm{E}(5+4X)$.

[2]

(b) Find $\mathrm{Var}(4-5X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=-0.8$. It is given that X and Y are independent.

(c) Find $\mathrm{E}(3X-7Y)$.

[2]

Solution

(a) $\mathrm{E}(5+4X)=5+4\mathrm{E}(X)$

$\mathrm{E}(5+4X)=5+4(1.6)$ (A1) for substitution

$\mathrm{E}(5+4X)=11.4$ A1

[2]

(b) $\mathrm{Var}(4-5X)=(-5)^2\mathrm{Var}(X)$

$\mathrm{Var}(4-5X)=(25)(0.25)$ (A1) for substitution

$\mathrm{Var}(4-5X)=6.25$ A1

[2]

(c) $\mathrm{E}(3X-7Y)=3\mathrm{E}(X)-7\mathrm{E}(Y)$

$\mathrm{E}(3X-7Y)=3(1.6)-7(-0.8)$ (A1) for substitution

$\mathrm{E}(3X-7Y)=10.4$ A1

[2]

Exercise 64

1. The random variable X is defined such that $\mathrm{E}(X) = 21.5$ and $\mathrm{Var}(X) = 3$.

(a) Find $\mathrm{E}(1 - 10X)$.

[2]

(b) Find $\mathrm{Var}(5 + 2X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y) = 20$. It is given that X and Y are independent.

(c) Find $\mathrm{E}(5 + 4X + 3Y)$.

[2]

2. The random variable X is defined such that $\mathrm{E}(5X) = 40$ and $\mathrm{Var}(1 + 2X) = 4$.

(a) Find $\mathrm{E}(X)$.

[2]

(b) Find $\mathrm{Var}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y) = 20$. It is given that X and Y are independent.

(c) Find $\mathrm{E}(-X - Y)$.

[2]

3. The random variable X is defined such that $\mathrm{E}(X) = -5$ and $\mathrm{Var}(X) = 16$.

(a) Find $\mathrm{E}(100 - X)$.

[2]

(b) Find $\mathrm{Var}(99 - 5X)$.

[2]

Another random variable Y is defined such that $\mathrm{Var}(Y) = 8$. It is given that X and Y are independent.

(c) Find $\mathrm{Var}(6X - 5Y)$.

[2]

4. The random variable X is defined such that $\mathrm{E}(8-7X)=29$ and $\mathrm{Var}(-7X)=147$.

(a) Find $\mathrm{E}(X)$.

[2]

(b) Find $\mathrm{Var}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{Var}(Y)=4.5$. It is given that X and Y are independent.

(c) Find $\mathrm{Var}(10Y-3X)$.

[2]

65 Paper 1 – Problems Involving Normal Distribution

Example

In a school, the male students' weights are normally distributed with mean 60 kg and standard deviation 3 kg, and the female students' weights are normally distributed with mean 50 kg and standard deviation 3.5 kg. Three male students are randomly chosen. Let X be the total weight of the selected students.

(a) Write down

(i) the mean of X;

(ii) the variance of X.

[2]

One male student and two female students are randomly chosen. Let Y be the total weight of the selected students.

(b) Write down

(i) the mean of Y;

(ii) the standard deviation of Y.

[2]

(c) Hence, find $P(Y > 170)$.

[2]

Solution

(a) (i) 180 kg A1

(ii) 27 kg A1

[2]

(b) (i) 160 kg A1

(ii) 5.79 kg A1

[2]

(c) $Y \sim N(160, 33.5)$

$P(Y > 170) = 0.0420176687$ (A1) for correct value

$P(Y > 170) = 0.0420$ A1

[2]

Exercise 65

1. In a supermarket, the weights of apples are normally distributed with mean 160 g and standard deviation 8 g, and the weights of oranges are normally distributed with mean 220 g and standard deviation 12 g. Four apples are randomly chosen. Let X be the total weight of the selected apples.

(a) Write down

(i) the mean of X;

(ii) the variance of X.

[2]

Two apples and two oranges are randomly chosen. Let Y be the total weight of the selected fruits.

(b) Write down

(i) the mean of Y;

(ii) the standard deviation of Y.

[2]

(c) Hence, find $\mathrm{P}(Y < 795)$.

[2]

2. In a tutorial class, the male students' heights are normally distributed with mean 178 cm and standard deviation 3 cm, and the female students' heights are normally distributed with mean 165 cm and standard deviation 4 cm. Six male students are randomly chosen. Let X be the total height of the selected students.

(a) Write down

(i) the exact value of the mean of X;

(ii) the variance of X.

[2]

(b) Hence, find $\mathrm{P}(1070 < X < 1090)$.

[2]

One male student and one female student are randomly chosen. Let Y be the total height of the two selected students.

(c) Find $\mathrm{P}(Y > 330)$.

[3]

3. In a snack shop, the weights of candy boxes are normally distributed with mean 500 g and standard deviation 13 g, and the weights of chocolate boxes are normally distributed with mean 470 g and standard deviation 17 g. Let X and Y be the weight of a candy box and a chocolate box respectively.

A candy box and a chocolate box are randomly chosen.

(a) Find $P(X+Y>1000)$.

[3]

(b) Find the probability that the candy box is heavier than the chocolate box.

[3]

15

4. In a town, the waiting times for buses are normally distributed with mean 10 minutes and standard deviation 1.5 minutes, and the waiting times for taxis are normally distributed with mean 7 minutes and standard deviation 0.75 minutes. Let X and Y be the waiting time for a bus and a taxi respectively, on a particular day.

(a) Find $P(16.2<X+Y<17.8)$.

[3]

(b) Sam starts his journey on a particular day and he needs to take either a bus or a taxi. Find the probability that a bus arrives earlier than a taxi.

[3]

66 Paper 1 – Problems Involving Poisson Distribution

Example

Two independent random variables X and Y are distributed such that $X \sim \text{Po}(9)$ and $Y \sim \text{Po}(7)$. Find

(a) $\text{E}(X+Y)$;

[2]

(b) $\text{P}(X+Y>16)$;

[2]

(c) $\text{Var}(3X+2Y)$.

[2]

Solution

(a) $\text{E}(X+Y)=\text{E}(X)+\text{E}(Y)$

$\text{E}(X+Y)=9+7$ (A1) for substitution

$\text{E}(X+Y)=16$ A1

[2]

(b) $\text{P}(X+Y>16)=1-\text{P}(X+Y\le 16)$ (M1) for valid approach

$\text{P}(X+Y>16)=1-0.5659624252$

$\text{P}(X+Y>16)=0.4340375748$

$\text{P}(X+Y>16)=0.434$ A1

[2]

(c) $\text{Var}(3X+2Y)=3^2\,\text{Var}(X)+2^2\,\text{Var}(Y)$

$\text{Var}(3X+2Y)=9(9)+4(7)$ (A1) for substitution

$\text{Var}(3X+2Y)=109$ A1

[2]

Exercise 66

1. Two independent random variables X and Y are distributed such that $X \sim \text{Po}(5.5)$ and $Y \sim \text{Po}(3.5)$. Find

(a) $\text{E}(X+Y)$;

[2]

(b) $\text{P}(5 < X+Y < 8)$;

[2]

(c) $\text{Var}(X-4Y)$.

[2]

2. Three independent random variables W, X and Y are distributed such that $W \sim \text{Po}(12)$, $X \sim \text{Po}(18)$ and $Y \sim \text{Po}(4)$. Find

(a) $\text{E}(W+X+Y)$;

[2]

(b) $\text{P}(W+X+Y \geq 30)$;

[2]

(c) $\text{Var}(3W-2X-Y)$.

[2]

3. In a fast food shop, the number of hamburgers and onion ring boxes sold follow Poisson distributions with means 24 per hour and 15.5 per hour respectively.

(a) Find the expected total number of hamburgers and onion ring boxes sold in a particular hour.

[2]

(b) Find the probability that a total number of 38 hamburgers and onion ring boxes are sold in a particular hour.

[2]

(c) Find the probability that, in six 1-hour intervals, a total number of 38 hamburgers and onion ring boxes are sold in each interval.

[2]

4. On a website, the number of views of a football video and a basketball video follow Poisson distributions with means 195 per day and 180 per day respectively.

(a) Find the expected total number of views of the football video and the basketball video on a particular day.

[2]

(b) Find the probability that there are a total of 376 views of the football video and the basketball video on a particular day.

[2]

(c) Find the probability that, in three 1-day periods, there are a total of 376 views of the football video and the basketball video on each period.

[2]

67 Paper 1 – General Problems

Example

The following table shows the probability distribution of a discrete random variable X.

x	10	20	30	40
$\text{P}(X=x)$	0.1	0.3	0.5	0.1

(a) Find $\text{E}(X)$.

[2]

Another random variable Y is defined such that $\text{E}(Y)=\text{Var}(Y)=15$. It is given that X and Y are independent, and $\text{Var}(X)=64$.

(b) Find $\text{E}(X+Y)$.

[2]

(c) Find $\text{Var}(3X+7Y)$.

[2]

Solution

(a) $\text{E}(X)=(10)(0.1)+(20)(0.3)+(30)(0.5)+(40)(0.1)$ (A1) for substitution

$\text{E}(X)=26$ A1

[2]

(b) $\text{E}(X+Y)=\text{E}(X)+\text{E}(Y)$

$\text{E}(X+Y)=26+15$ (A1) for substitution

$\text{E}(X+Y)=41$ A1

[2]

(c) $\text{Var}(3X+7Y)=3^2\text{Var}(X)+7^2\text{Var}(Y)$

$\text{Var}(3X+7Y)=9(64)+49(15)$ (A1) for substitution

$\text{Var}(3X+7Y)=1311$ A1

[2]

Exercise 67

1. The following table shows the probability distribution of a discrete random variable X.

x	2	4	6	8
$\mathrm{P}(X=x)$	0.6	0.1	0.15	0.15

(a) Find $\mathrm{E}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=10$ and $\mathrm{Var}(Y)=2$. It is given that X and Y are independent, and $\mathrm{Var}(X)=5.31$.

(b) Find $\mathrm{E}(5X+2Y)$.

[2]

(c) Find $\mathrm{Var}(5X+2Y)$.

[2]

2. The random variable X is defined such that $X \sim \mathrm{B}(100, 0.65)$.

(a) Find $\mathrm{E}(X)$.

[2]

(b) Find $\mathrm{Var}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=30$ and $\mathrm{Var}(Y)=7$. It is given that X and Y are independent.

(c) Find $\mathrm{E}(2X-7Y)$.

[2]

(d) Find $\mathrm{Var}(2X-7Y)$.

[2]

3. A factory makes lamps. The probability that a lamp is defective is 0.03. The factory tests a random sample of 15 lamps. Let X be the number of defective lamps in the sample.

(a) Find $\mathrm{E}(X)$.

[2]

(b) Find $\mathrm{Var}(X)$.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=-1.2$ and $\mathrm{Var}(Y)=0.8$. It is given that X and Y are independent.

(c) Find $\mathrm{E}(-X-Y)$.

[2]

(d) Find $\mathrm{Var}(-X-Y)$.

[2]

4. A box contains four yellow balls and one purple ball. Chester plays a game by randomly drawing a ball one by one from the box, without replacement. The game ends when a purple ball is being drawn.

Chester gets \$20 and \$10 if he draws the purple ball in his first trial and second trial respectively. He gets nothing if he draws the purple ball after the second trial. Let X be the amount of payoff to Chester.

(a) (i) Write down $\mathrm{P}(X=20)$.

(ii) Find $\mathrm{P}(X=0)$.

(iii) Hence, find $\mathrm{E}(X)$.

[5]

Another random variable Y is defined such that $\mathrm{E}(X+6Y)=8$. It is given that X and Y are independent.

(b) Find $\mathrm{E}(Y)$.

[2]

Chapter

16

Point Estimation

SUMMARY POINTs

✓ Central limit theorem:

1. $X_i \sim N(\mu, \sigma^2)$
2. $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_n}{n}$: Sample mean
3. $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{n}\right)$ when n is sufficiently large

✓ Properties of point estimation:

1. $s_{n-1}^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$: Sample variance
2. $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2$, where $s_n^2 = \dfrac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n}$
3. $\mathrm{E}(\bar{X}) = X$
4. $\mathrm{E}(s_{n-1}^2) = \sigma^2$

Solutions of Chapter 16

68 Paper 1 – Central Limit Theorem

Example

The random variable X is defined such that $\mathrm{E}(X)=20$ and $\mathrm{Var}(X)=6$. A random sample of 50 observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. By using the central limit theorem,

(a) write down $\mathrm{E}(\bar{X})$;

[1]

(b) find $\mathrm{Var}(\bar{X})$;

[2]

(c) find $\mathrm{P}(\bar{X}>19.7)$.

[2]

Solution

(a) 20 — A1

[1]

(b) $\mathrm{Var}(\bar{X})=\frac{\mathrm{Var}(X)}{n}$

$\mathrm{Var}(\bar{X})=\frac{6}{50}$ — (A1) for substitution

$\mathrm{Var}(\bar{X})=\frac{3}{25}$ — A1

[2]

(c) $\bar{X}\sim \mathrm{N}\left(20,\frac{3}{25}\right)$ — (M1) for valid approach

$\mathrm{P}(\bar{X}>19.7)=0.8067619307$

$\mathrm{P}(\bar{X}>19.7)=0.807$ — A1

[2]

Exercise 68

1. The random variable X is defined such that $E(X)=300$ and $Var(X)=4.5$. A random sample of 180 observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. By using the central limit theorem,

(a) write down $E(\bar{X})$;

[1]

(b) find $Var(\bar{X})$;

[2]

(c) find $P(\bar{X}<299.85)$.

[2]

2. The random variable X is defined such that $E(X)=-2$ and $Var(X)=8$. A random sample of 32 observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. By using the central limit theorem,

(a) write down $E(\bar{X})$;

[1]

(b) find the standard deviation of $\bar{X}$;

[2]

(c) find $P(|\bar{X}|<1.5)$.

[3]

3. The random variable X is defined such that $E(X)=5$ and $Var(X)=1.6$. A random sample of 50 observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. By using the central limit theorem,

(a) write down $E(\bar{X})$;

[1]

(b) find $Var(\bar{X})$.

[2]

Another random variable Y is defined such that $E(Y)=-5$ and $Var(Y)=0.8$. A new random sample of 50 observations is selected from the distribution of Y. Let $\bar{Y}$ be the mean of the new sample. It is given that $\bar{X}$ and $\bar{Y}$ are independent.

(c) Find $P(\bar{X}+\bar{Y}>0.1)$.

[3]

4. The random variable X is defined such that $\mathrm{E}(X)=180$ and $\mathrm{Var}(X)=12$. A random sample of n observations is selected from the distribution of X. Let $\bar{X}$ be the mean of the sample. It is given that the standard deviation of $\bar{X}$ is $\frac{1}{3}$.

(a) By using the central limit theorem, find n.

[2]

Another random variable Y is defined such that $\mathrm{E}(Y)=160$ and $\mathrm{Var}(Y)=48$. A new random sample of 216 observations is selected from the distribution of Y. Let $\bar{Y}$ be the mean of the new sample. It is given that $\bar{X}$ and $\bar{Y}$ are independent.

(b) Find $\mathrm{P}(9<\bar{X}-\bar{Y}<21)$.

[3]

69 Paper 1 – Unbiased Estimates

Example

The number of breakdowns on a road in a city is studied. A sample of 20 observations is recorded on 20 different days. Let X be the number of breakdowns on a particular day. It is given that $s_n^2 = 3.8$.

(a) Find s_{n-1}^2.

[2]

Let $\bar{X}$ be the mean of the sample. It is also given that there are 60 breakdowns in these 20 days.

(b) Find an unbiased estimate of the population mean.

[2]

Solution

(a) $s_{n-1}^2 = \frac{n}{n-1} s_n^2$

$s_{n-1}^2 = \frac{20}{20-1}(3.8)$ (A1) for substitution

$s_{n-1}^2 = 4$ A1

[2]

(b) An unbiased estimate

$= \bar{X}$ (A1) for correct approach

$= \frac{60}{20}$

$= 3$ A1

[2]

Exercise 69

1. The number of products produced by a machine operator is studied. A sample of 30 observations is recorded on 30 different weeks. Let X be the number of products produced by the machine operator in a particular week. It is given that $s_n = 5$.

(a) Find s_{n-1}.

[2]

Let $\bar{X}$ be the mean of the sample. It is also given that there are 1770 products produced by the machine operator in these 30 weeks.

(b) Find an unbiased estimate of the population mean.

[2]

2. The number of complaint calls in a customer service department is studied. A sample of 25 observations is recorded on 25 different working days. It is given that $s_{n-1}^2 = 1.25$.

(a) Find s_n^2.

[2]

It is also given that an unbiased estimate of the population mean is 7.8.

(b) Find the total number of complaint calls in the department in these 25 working days.

[2]

3. A random sample of size 60 is taken from a population. It is given that $s_{n-1} = 2.37$.

(a) Find s_n.

[2]

It is also given that an unbiased estimate of the population mean is -3.8.

(b) Find the total sum of numbers in the sample.

[2]

4. An investigation was carried out into the number of seconds that cargo trains are late in departing. A random sample of n cargo trains is observed. Let X be the number of seconds that a cargo train is late in departing. It is given that $\bar{X} = 16.6$, $s_n^2 = 3.27$ and $s_{n-1}^2 = 3.3$.

(a) Write down an unbiased estimate of

(i) the population mean;

(ii) the population variance.

[2]

(b) Find n.

[3]

16

Chapter 17

Interval Estimation

SUMMARY POINTs

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is known:

1. $\bar{X}=\dfrac{X_1+X_2+\cdots+X_n}{n}$: Sample mean

2. $\left(\bar{X}-z_{\frac{\alpha}{2}}\cdot\dfrac{\sigma}{\sqrt{n}},\bar{X}+z_{\frac{\alpha}{2}}\cdot\dfrac{\sigma}{\sqrt{n}}\right)$, where $P\left(Z>z_{\frac{\alpha}{2}}\right)=\dfrac{\alpha}{2}$

3. $2z_{\frac{\alpha}{2}}\cdot\dfrac{\sigma}{\sqrt{n}}$: Interval width

✓ $(1-\alpha)\%$ confidence interval for population mean μ when the population variance σ^2 is unknown:

1. $\left(\bar{X}-t_{n-1,\frac{\alpha}{2}}\cdot\dfrac{s_{n-1}}{\sqrt{n}},\bar{X}+t_{n-1,\frac{\alpha}{2}}\cdot\dfrac{s_{n-1}}{\sqrt{n}}\right)$, where $P\left(\bar{X}>t_{n-1,\frac{\alpha}{2}}\right)=\dfrac{\alpha}{2}$

2. $2t_{n-1,\frac{\alpha}{2}}\cdot\dfrac{s_{n-1}}{\sqrt{n}}$: Interval width

Solutions of Chapter 17

70 Paper 1 – Confidence Intervals

Example

The noon temperature, in degree Celsius, in a city through the year is studied. A sample of 10 observations is recorded on 10 different days, recorded as below:

18.5, 19.4, 20.1, 22.3, 19.5, 24.1, 23.4, 17.9, 21.5, 23.3

(a) Find an unbiased estimate of the population mean.

[2]

(b) Find s_{n-1}.

[2]

(c) Construct a 95% confidence interval for the population mean, giving the answer to three decimal places.

[2]

17

Solution

(a) An unbiased estimate

$$=\frac{18.5+19.4+\cdots+23.3}{10}$$ (A1) for correct approach

$=21\ ^{\circ}\mathrm{C}$ A1

[2]

(b) $$s_{n-1}=\sqrt{\frac{(18.5-21)^2+(19.4-21)^2+\cdots+(23.3-21)^2}{10-1}}$$ (A1) for correct approach

$s_{n-1}=2.213092356\ ^{\circ}\mathrm{C}$

$s_{n-1}=2.21\ ^{\circ}\mathrm{C}$ A1

[2]

(c) 95% confidence interval:

(19.417, 22.583) A2

[2]

Exercise 70

1. The weight of a margarine block, in grams, produced in a production line is studied. A sample of 12 observations is recorded as below:

300, 302, 308, 305, 297, 301, 504, 501, 495, 498, 510, 491

(a) Find an unbiased estimate of the population mean.

[2]

(b) Find s_{n-1}.

[2]

(c) Construct a 99% confidence interval for the population mean, giving the answer to two decimal places.

[2]

2. The time to install a program, in seconds, for a group of computers is studied. A sample of 6 observations is recorded as below:

66, 50, 34, 51, 71, 52

(a) Find an unbiased estimate of the population mean.

[2]

It is given that the population variance is $\sigma^2 = 1.01\,\text{s}^2$.

(b) Construct a 95% confidence interval for the population mean, giving the answer to three decimal places.

[2]

3. Suki frequently travels from home to cinema. On n randomly selected occasions the journey time t minutes was recorded. It is given that $\bar{t} = 20$ minutes, $\sigma = 3$ minutes and the total time used in these n journeys is 500 minutes.

(a) Find n.

[2]

(b) Construct a 90% confidence interval for the population mean, giving the answer to three decimal places.

[2]

(c) Hence, write down the interval width of the confidence interval, giving the answer to three decimal places.

[1]

4. The weight w of a bronze container, in kilograms, produced in a factory is studied. A random sample of n containers was weighed. It is given that $\bar{w} = 70$ kg, $s_{n-1} = 3$ kg and $s_{n-1}^2 = \frac{16}{15} s_n^2$.

(a) Find n.

[3]

(b) Construct a 90% confidence interval for the population mean, giving the answer to three decimal places.

[2]

(c) Hence, write down the interval width of the confidence interval, giving the answer to two decimal places.

[1]

71 Paper 1 – Standard Deviations

Example

The height of an orchid, in centimetres, from a store is studied. A random sample of 20 orchids is measured. It is given that the 95% confidence interval for the population mean is $(21.75, 22.35)$.

(a) Find an unbiased estimate of the population mean.

[2]

Let σ^2 be the known population variance.

(b) Find σ.

[3]

Solution

(a) An unbiased estimate

$= \bar{X}$ (A1) for correct approach

$= \dfrac{21.75 + 22.35}{2}$

$= 22.05 \text{ cm}$ A1

[2]

(b) $22.35 - 21.75 = 2(1.959963986)\left(\dfrac{\sigma}{\sqrt{20}}\right)$ M1A1

$\sigma = 0.6845231831 \text{ cm}$

$\sigma = 0.685 \text{ cm}$ A1

[3]

Exercise 71

1. The Body Mass Index (BMI) of a group of patients in a ward is studied. 8 patients are randomly selected and their BMIs are measured. It is given that the 99% confidence interval for the population mean is $(18.25, 23.05)$.

(a) Find an unbiased estimate of the population mean.

[2]

Let σ^2 be the known population variance.

(b) Find σ.

[3]

2. The number of goals in a league in the past few years is studied. 25 games are randomly selected and the numbers of goals are recorded. It is given that the 95% confidence interval for the population mean is $(2.995, 3.365)$.

(a) Find an unbiased estimate of the population mean.

[2]

(b) Find s_{n-1}.

[3]

3. The length of a rope, in centimetres, sold in a store is studied. 16 ropes are randomly selected and the corresponding lengths are measured. It is given that the sample mean is 37.5 cm and the width of the 95% confidence interval for the population mean is 2.2 cm.

(a) Explain why the 90% confidence interval for the population mean is a subset of the 95% confidence interval for the population mean.

[1]

(b) Write down the 95% confidence interval for the population mean.

[1]

(c) Find s_{n-1}.

[3]

4. The width of a board, in centimetres, sold in a bazaar is studied. 9 boards are randomly selected and the corresponding widths are measured. It is given that the sample mean is 198 cm and the width of the 99% confidence interval for the population mean is 12.8 cm.

(a) Explain why the 95% confidence interval for the population mean is a subset of the 99% confidence interval for the population mean.

[1]

(b) Write down the 99% confidence interval for the population mean.

[1]

Let σ^2 be the known population variance.

(c) Find σ.

[3]

Chapter

18

Bivariate Analysis

SUMMARY POINTs

✓ Coefficient of determination:

R^2 : Coefficient of determination

R^2 % of the variability of the data can be explained by the regression model

✓ Sum of square residuals:

x	x_1	x_2	...	x_n
y	y_1	y_2	...	y_n
Predicted value of y	$\hat{y}_1$	$\hat{y}_2$	...	$\hat{y}_n$

$SS_{res} = \frac{1}{n}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2$: Sum of square residuals

✓ Non-linear regressions:

1. Quadratic regression
2. Cubic regression
3. Quartic regression
4. Exponential regression
5. Logarithmic regression
6. Power regression
7. Logistic regression

Solutions of Chapter 18

72 Paper 1 – Coefficient of Determination

Example

The evaporation rates of six different chemical solutions are studied. Six different chemical solutions are stored in six different beakers. The storage time x (in days) and the corresponding loss in evaporation y (in mL) are shown in the following table.

Storage time (x days)	5	7	9	12	13	13
Loss in evaporation (y mL)	8	8	7	11	21	18

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Hence, estimate the loss in evaporation of a solution when it is stored for 10 days.

[4]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

Solution

(a) (i) $a = 1.410557185$

$a = 1.41$ A1

$b = -1.703812317$

$b = -1.70$ A1

(ii) The estimated loss

$= 1.410557185(10) - 1.703812317$ (A1) for substitution

$= 12.40175953$ mL

$= 12.4$ mL A1

[4]

(b) (i) $r = 0.8042297654$
$r = 0.804$ A1

(ii) $R^2 = 0.6467855156$
$R^2 = 0.647$ A1

(iii) 64.7% of the variability of the data is explained by the regression model. A1

[3]

Exercise 72

1. The following table shows the age of a car x (in years) and the corresponding distance travelled y (in km).

Age of a car (x years)	3	5	7	9	10	10
Distance travelled (y km)	26000	28000	49000	47000	58000	82000

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b, giving the answers correct to the nearest integer.

(ii) Hence, estimate the distance travelled of a car when its age is 4.5 years.

[4]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

2. The following table shows the age of a silver coin x (in years) and the corresponding pure weight y (in g).

Age of a silver coin (x years)	13	16	19	23	28
Pure weight (y g)	6	7	5	4.8	4.6

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Hence, estimate the pure weight of a silver coin when it is a quarter-century old.

[4]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

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3. The expenditures of a group of market customers are studied. The following table shows the times x (in minutes) of six customers spent in a market and their corresponding expenditures y (in USD).

Time spent (x minutes)	47	28	35	7	11	50
Expenditure (y USD)	50	50	42	16	5	15

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Explain what the gradient a represents.

[3]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

4. The following table shows the annual rainfall x (in cm) of a region in 5 different years and the corresponding annual corn yield rate y (in $\%\text{yr}^{-1}$).

Annual rainfall (x cm)	57	55	53	51	49
Annual corn yield rate ($y\ \%\text{yr}^{-1}$)	6	5.9	5.4	5.4	5.15

(a) The relationship between the variables is modelled by the regression equation $y = ax + b$.

(i) Write down the value of a and of b.

(ii) Explain what the intercept b represents.

[3]

(b) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

73 Paper 1 – Sum of Square Residuals

Example

The distances travelled of a group of buses are studied. The following table shows the distances travelled x (in km) of five buses and the corresponding amounts of fuel required y (in L).

Distance travelled (x km)	100	105	140	153	180
Amount of fuel required (y L)	32	38	46	50	60

It is suggested that the relationship between the variables can be modelled by the regression equation $y = 0.3x + 2$.

(a) Use the suggested model to write down the estimated amount of fuel required when a bus is travelled by

(i) 100 km;

(ii) 105 km;

(iii) 153 km.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals.

[2]

It is given that the coefficient of determination of this model is 0.976.

(c) Interpret the coefficient of determination.

[1]

Solution

(a) (i) 32 L A1

(ii) 33.5 L A1

(iii) 42.5 L A1

[3]

18

(b) $SS_{res} = (32-32)^2 + (33.5-38)^2$
$+((0.3(140)+2)-46)^2 + (42.5-50)^2$ (A1) for correct approach
$+((0.3(180)+2)-60)^2$
$SS_{res} = 84.5$ A1

[2]

(c) 97.6% of the variability of the data is explained by the regression model. A1

[1]

Exercise 73

1. The monthly rent prices of several flats are studied. The following table shows the distances x (in km) of five flats from the central railway station and the corresponding monthly rent prices y (in USD).

Distance (x km)	0.2	0.5	0.8	1.2	1.7
Monthly rent price (y USD)	1100	900	850	500	530

It is suggested that the relationship between the variables can be modelled by the regression equation $y = -400x + 1130$.

(a) Use the suggested model to write down the estimated monthly rent price of a flat when the distance between the flat and the central railway station is

(i) 0.2 km;

(ii) 0.8 km;

(iii) 1.7 km.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals.

[2]

It is given that the coefficient of determination of this model is 0.872.

(c) Interpret the coefficient of determination.

[1]

2. The following table shows the times x (in hours) of six students spending on revision and the corresponding test results y (in marks).

Revision time (x hours)	1.5	3.5	5	5.5	6	7.5
Test result (y marks)	60	75	85	75	70	75

It is suggested that the relationship between the variables can be modelled by the regression equation $y=-1.15x^2+12.5x+45$.

(a) Use the suggested model to write down the estimated test result of a student when the revision time spent by the student is

(i) 3.5 hours;

(ii) 5.5 hours;

(iii) 7.5 hours,

giving the answers in exact values.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals.

[2]

It is given that the coefficient of determination of this model is 0.632.

(c) Interpret the coefficient of determination.

[1]

3. The lengths and weights of a group of salmons are studied. The following table shows the lengths x (in cm) of three salmons and the corresponding weights y (in g).

Length (x cm)	70	80	90
Weight (y g)	6700	7400	7500

Two different models are suggested to model the relationship between the variables. The following table shows the estimated weights under two models:

	Weight (y g) estimated by	
	Model 1	Model 2
Length (x cm)	$y = 40x + 4000$	$y = 4600 \cdot 1.01^x$
70	6800	a
80	b	$4600 \cdot 1.01^{80}$
90	7600	c

(a) Write down the values of

(i) a;

(ii) b;

(iii) c,

giving the answers correct to the nearest integer.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals of the Model 1.

[2]

The model with the smaller sum of square residuals is selected. It is given that the sum of square residuals of the Model 2 is 28400000.

(c) Determine which model is selected.

[1]

4. The following table shows the lengths x (in cm) of the right feet of three students and the corresponding heights y (in cm).

Right foot length (x cm)	26	24	22
Height (y cm)	140	130	135

Two different models are suggested to model the relationship between the variables. The following table shows the estimated heights under two models:

	Height (y cm) estimated by	
	Model 1	Model 2
Right foot length (x cm)	$y = 2x^2 - 90x + 1200$	$y = 100 \cdot 1.02^x$
26	a	$100 \cdot 1.02^{26}$
24	b	$100 \cdot 1.02^{24}$
22	188	c

(a) Write down the values of

(i) a;

(ii) b;

(iii) c,

giving the answers correct to the nearest integer.

[3]

(b) Hence, calculate SS_{res}, the sum of square residuals of the Model 2.

[2]

The model with the smaller sum of square residuals is selected. It is given that the sum of square residuals of the Model 1 is 11837.

(c) Determine which model is selected.

[1]

18

74 Paper 1 – Non-Linear Regressions

Example

The noon temperatures of some towns are studied. The following table shows the altitude x (in m) of five towns and the corresponding noon temperature y (in $^{\circ}\text{C}$).

Altitude (x m)	50	100	150	200	250
Noon temperature (y $^{\circ}$C)	24	26	32	33	32

It is suggested that the relationship between the variables can be modelled by the regression equation $y = ax^2 + bx + c$, where a, b, $c \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the noon temperature, giving the coefficients correct to 3 significant figures if necessary.

(ii) Hence, estimate the noon temperature of a town of altitude 125 m.

[4]

(b) (i) Write down the coefficient of determination.

(ii) Hence, interpret the coefficient of determination.

[2]

Solution

(a) (i) $y = -0.000371x^2 + 0.153x + 17$ A2

(ii) The estimated noon temperature

$= -0.0003714285714(125)^2$
$+ 0.1534285714(125) + 17$ (A1) for substitution

$= 30.375\ ^{\circ}\text{C}$

$= 30.4\ ^{\circ}\text{C}$ A1

[4]

(b) (i) $R^2 = 0.9889336016$

$R^2 = 0.989$ A1

(ii) 98.9% of the variability of the data is explained by the regression model. A1

[2]

Exercise 74

1. The heights and weights of a group of children are studied. The following table shows the heights x (in cm) of five children and the corresponding weights y (in kg).

Height (x cm)	140	145	150	155	160
Weight (y kg)	55	61	58	56	58

It is suggested that the relationship between the variables can be modelled by the regression equation $y = ax^3 + bx^2 + cx + d$, where a, b, c, $d \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the weight of children, giving the coefficients correct to 3 significant figures if necessary.

(ii) Hence, estimate the weight of a child of height 148 cm.

[4]

(b) (i) Write down the coefficient of determination.

(ii) Hence, interpret the coefficient of determination.

[2]

18

2. The ages and weights of a group of turtles are studied. The following table shows the ages x (in years old) of four turtles and the corresponding weights y (in g).

Age (x years old)	0.5	1	1.5	2
Weight (y g)	20	25	30	23

It is suggested that the relationship between the variables can be modelled by the regression equation $y = ax^b$, where a, $b \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the weight of turtles, giving the coefficients correct to 3 significant figures if necessary.

(ii) Hence, estimate the weight of a turtle of age 1.25 years old.

[4]

(b) (i) Write down the coefficient of determination.

(ii) Hence, interpret the coefficient of determination.

[2]

3. The performance of a football team is studied. The following table shows the number of goals scored x of the team in five matches and the number of goals conceded y in the matches.

Number of goals scored (x)	1	2	1	2	3
Number of goals conceded (y)	1	4	2	5	8

It is suggested that the relationship between the variables can be modelled by the regression equation $y = ax^2 + bx + c$, where a, b, $c \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the number of goals conceded.

(ii) Write down the coefficient of determination.

[3]

(b) Write down SS_{res}, the sum of square residuals.

[2]

(c) By using $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$, find SS_{tot}, the total sum of squares.

[2]

4. The relationship between the cost and the revenue of the products in a company is studied. The following table shows the cost x (in EUR) of the four products in the company and the corresponding revenue y (in EUR).

Cost (x EUR)	3	4	5	6
Revenue (y EUR)	6	8	11	15

It is suggested that the relationship between the variables can be modelled by the regression equation $y = a \cdot b^x$, where a, $b \in \mathbb{R}$.

(a) (i) Write down the least square regression curve for the revenue of products.

(ii) Write down the coefficient of determination, giving the answer in five decimal places.

[3]

(b) Write down SS_{res}, the sum of square residuals.

[2]

(c) By using $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$, find SS_{tot}, the total sum of squares.

[2]

Chapter

19

Statistical Tests

SUMMARY POINTs

✓ More about χ^2 test for independence and χ^2 goodness of fit test:
All expected frequencies E_i should be greater than 5

✓ Z test when the population variance σ^2 is known:
μ: Population mean
H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

✓ One sample t test when the population variance σ^2 is unknown:
μ: Population mean
H_0: $\mu = \mu_0$
H_1: $\mu > \mu_0$, $\mu < \mu_0$ (for 1-tailed test), $\mu \neq \mu_0$ (for 2-tailed test)
H_0 is rejected if the p-value is less than the significance level
H_0 is not rejected if the p-value is greater than the significance level

SUMMARY POINTs

✓ Paired t test:

$d = x - y$: Difference between each pair of data from two variables x and y

H_0: $\mu_d = 0$

H_1: $\mu_d > 0$, $\mu_d < 0$ (for 1-tailed test), $\mu_d \neq 0$ (for 2-tailed test)

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving binomial distribution:

$X \sim \mathrm{B}(n, p)$

H_0: $p = p_0$

H_1: $p > p_0$, $p < p_0$

x: Observed value of X

$\mathrm{P}(X \geq x)$ for H_1: $p > p_0$ or $\mathrm{P}(X \leq x)$ for H_1: $p < p_0$: p-value under $X \sim \mathrm{B}(n, p_0)$

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ Test involving Poisson distribution:

$X \sim \mathrm{Po}(\lambda)$

H_0: $\lambda = \lambda_0$

H_1: $\lambda > \lambda_0$, $\lambda < \lambda_0$

x: Observed value of X

$\mathrm{P}(X \geq x)$ for H_1: $\lambda > \lambda_0$ or $\mathrm{P}(X \leq x)$ for H_1: $\lambda < \lambda_0$: p-value under $X \sim \mathrm{Po}(\lambda_0)$

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

SUMMARY POINTs

✓ Test involving bivariate normal distribution:

ρ : Product moment correlation coefficient

H_0: $\rho = 0$

H_1: $\rho > 0$, $\rho < 0$ (for 1-tailed test), $\rho \neq 0$ (for 2-tailed test)

H_0 is rejected if the p-value is less than the significance level

H_0 is not rejected if the p-value is greater than the significance level

✓ Type I and type II errors:

α : Significance level

$\mathrm{P}(\text{Reject } H_0 \mid H_0 \text{ is true}) = \alpha$: Type I error

$\mathrm{P}(\text{Not reject } H_0 \mid H_0 \text{ is not true})$: Type II error

Solutions of Chapter 19

75 Paper 1 – Goodness-Of-Fit Test

Example

In an experiment, an unbiased six-sided die with faces 0, 1, 2, 3, 4 and 5 is tossed for 100 times. The following table shows the frequencies of the outcomes:

Outcome	0	1	2	3	4	5
Frequency	6	15	22	29	13	15

A χ^2 goodness of fit test is conducted at a 5% significance level to determine whether the data can be modelled by Poisson distribution with mean 3.

(a) Write down the null hypothesis of the test.

[1]

(b) (i) Write down the value of the expected frequency of the outcome "0".

(ii) Hence, write down the degree of freedom of the test.

[2]

(c) Find the value of χ^2_{calc}, the test statistic.

[2]

The critical value is given by 9.488.

(d) State the conclusion of the test with a reason.

[2]

Solution

(a) H_0: The data follows a Poisson distribution with mean 3. A1

[1]

(b) (i) 4.97 A1

(ii) 4 A1

[2]

(c) 3.52 A2

[2]

(d) The null hypothesis is not rejected. A1
As $\chi^2_{calc} < 9.488$. A1

[2]

Exercise 75

1. The following table shows the number of accidents on a highway in the last 120 days:

Number of accidents	0	1	2	3	4 or more
Number of days	40	25	40	13	2

A χ^2 goodness of fit test is conducted at a 10% significance level to determine whether the data can be modelled by Poisson distribution with mean 2.

(a) Write down the null hypothesis of the test.

[1]

(b) Write down the value of the expected frequency of the category "4 or more".

[1]

(c) Write down the degree of freedom of the test.

[1]

(d) Find the value of χ^2_{calc}, the test statistic.

[2]

The critical value is given by 7.779.

(e) State the conclusion of the test with a reason.

[2]

2. The following table shows the number of goals conceded by a goalkeeper in 100 matches:

Number of goals conceded	0	1	2	3	4	5
Frequency	4	18	35	21	17	5

A χ^2 goodness of fit test is conducted at a 5% significance level to determine whether the data can be modelled by Binomial distribution with parameters $\mathrm{B}(5, 0.4)$.

(a) Write down the null hypothesis of the test.

[1]

(b) (i) Write down the value of the expected frequency of the category "5".

(ii) Hence, write down the degree of freedom of the test.

[2]

(c) Find the value of χ^2_{calc}, the test statistic.

[2]

The critical value is given by 9.488.

(d) State the conclusion of the test with a reason.

[2]

3. The following table shows the inner diameters of 60 plastic tubes produced in a factory:

Inner diameter (mm)	$d < 10$	$10 \le d < 12$	$12 \le d < 14$	$d \ge 14$
Frequency	7	16	22	15

A χ^2 goodness of fit test is conducted at a 5% significance level to determine whether the data can be modelled by Normal distribution with parameters $\mathrm{N}(13, 4)$.

(a) Write down the null hypothesis of the test.

[1]

(b) (i) Write down the value of the expected frequency of the category "$d < 10$".

(ii) Hence, write down the degree of freedom of the test.

[2]

(c) Find the p-value.

[2]

(d) State the conclusion of the test with a reason.

[2]

4. The following table shows the weights of 300 cats living in a castle:

Weight (kg)	$w < 3$	$3 \le w < 3.5$	$3.5 \le w < 4$	$w \ge 4$
Frequency	49	131	83	37

A χ^2 goodness of fit test is conducted at a 1% significance level to determine whether the data can be modelled by Normal distribution with parameters $\mathrm{N}(3.4, 0.15)$.

(a) Write down the null hypothesis of the test.

[1]

(b) Write down the value of the expected frequency of the category "$w \ge 4$".

[1]

(c) Write down the degree of freedom of the test.

[1]

(d) Find the p-value.

[2]

(e) State the conclusion of the test with a reason.

[2]

76 Paper 1 – Z-Test

Example

The lengths of pencils produced in a factory is studied. It is given that the population standard deviation of the length is 0.1 cm. 50 pencils are randomly selected to form a sample such that the sample mean is 15.05 cm. Norman claims that the population mean μ is greater than 15 cm.

A hypothesis test is conducted at a 5% significance level to test Norman's claim.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

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Solution

(a) (i) $H_0: \mu = 15$ A1

(ii) $H_1: \mu > 15$ A1

[2]

(b) $p\text{-value} = 0.0002035177224$ (A1) for correct value

$p\text{-value} = 0.000204$ A1

[2]

(c) The null hypothesis is rejected. A1

As $p\text{-value} < 0.05$. A1

[2]

Exercise 76

1. The amount of chemical in a substance is being studied. It is given that the population standard deviation of the amount of chemical is 3.1 mg. 60 samples of the substances are randomly selected such that the sample mean of the amount of chemical is 16.2 mg. A hypothesis test is conducted at a 5% significance level to test whether the population mean μ differs from 17 mg.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

2. The numbers of chocolate balls in gift boxes is being studied. It is given that the population variance of the number of chocolate balls is 4. Eighty gift boxes are randomly selected to form a sample such that the sample mean number of chocolate balls in each box is 27.95. Gordon claims that the population mean number of chocolate balls μ is less than 28.

A hypothesis test is conducted at a 10% significance level to test Gordon's claim.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

3. The scores of a survey about the performance of a singer is being studied. It is given that the population variance of the scores is 3. 16 surveys are randomly selected to form a sample, as shown as follows:

7	9	8	5	3	6	10	10
10	8	7	10	4	9	7	2

A hypothesis test is conducted at a 5% significance level to test whether the population mean score μ is greater than 7.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) (i) Write down the sample mean.

(ii) Hence, find the test statistic z .

[3]

The critical value is given by 1.645.

(c) State the conclusion of the test with a reason.

[2]

4. The numbers of passengers passing a boarding gate is being studied. It is given that the population variance of the number of passengers is 28.

24 daily records of the number of passengers passing the boarding gate are randomly selected to form a sample, as shown as follows:

98	102	122	111	105	140	93	101
107	111	97	108	96	102	94	120
118	108	105	103	104	95	111	93

A hypothesis test is conducted at a 2% significance level to test whether the population mean number of passengers passing the boarding gate μ is different from 100, the target number of passengers in an advertisement.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) (i) Write down the sample mean.

(ii) Hence, find the test statistic z .

[3]

The null hypothesis is not rejected if $|z| < 2.326$.

(c) State the conclusion of the test with a reason.

[2]

77 Paper 1 – Paired t-Test

Example

In a library, the habit of reading books of young people is investigated. The table below shows the number of fiction books read by the same group of 8 people in two consecutive months. At the end of each month, they are required to complete a survey about the number of fiction books read.

Person	A	B	C	D	E	F	G	H
Number of fictions read in the first month (x)	6	8	4	2	1	3	2	1
Number of fictions read in the second month (y)	7	10	4	5	1	2	5	4

The library manager wants to investigate whether there is an increase in the number of fiction books read on average. A paired t-test is conducted at a 5% significance level. Define $d = x - y$.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) Write down the value of t, the test statistic.

[1]

(d) State the conclusion of the test with a reason.

[2]

Solution

(a) (i) $H_0: \mu_d = 0$ A1

(ii) $H_1: \mu_d < 0$ A1

[2]

(b) $p\text{-value} = 0.0225860627$ (A1) for correct value

$p\text{-value} = 0.0226$ A1

[2]

(c) -2.43 A1

[1]

(d) The null hypothesis is rejected. A1

As $p\text{-value} < 0.05$. A1

[2]

Exercise 77

1. The table below shows the typing speed (words per minute) of a group of 7 students in a vocational training school before and after training:

Student	A	B	C	D	E	F	G
Typing speed before training (x words per minute)	37	31	35	42	18	15	26
Typing speed after training (y words per minute)	33	49	59	42	34	25	42

The curriculum development officer of the school wants to investigate whether on average the students show improvement after training. A paired t-test is conducted at a 1% significance level. Define $d = x - y$.

(a) Write down the alternative hypothesis of the test.

[1]

(b) Find the p-value.

[2]

(c) Write down the value of t, the test statistic.

[1]

(d) State the conclusion of the test with a reason.

[2]

2. Two surveys are conducted to measure the students' satisfaction on the services provided by the sports centre. A score from 0 to 10 is used in the surveys, where 0 represents absolute dissatisfaction and 10 represents absolute satisfaction. The table below shows the results of the surveys completed by 8 students:

Student	A	B	C	D	E	F	G	H
Scores from the first survey (x)	8	10	8	9	10	6	3	3
Scores from the second survey (y)	6	10	7	5	9	6	5	3

The student union committee members want to investigate whether the mean scores of the second survey is lower. A paired t-test is conducted at a 5% significance level. Define $d = x - y$.

(a) Write down the alternative hypothesis of the test.
[1]

(b) Find the p-value.
[2]

(c) Write down the value of t, the test statistic.
[1]

(d) State the conclusion of the test with a reason.
[2]

3. A teacher wants to know whether the supplementary lessons between the mock examination and the official examination can improve students' mean score in the official examination. The table below shows the scores (out of 100) of 9 students in two examinations:

Student	A	B	C	D	E	F	G	H	I
Mock exam score (x)	45	55	83	79	82	62	67	80	75
Official exam score (y)	37	52	61	90	94	96	87	71	81

An appropriate test is conducted at a 10% significance level.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.
[2]

(b) Find the p-value.
[2]

(c) State the conclusion of the test with a reason.

[2]

One student is randomly selected.

(d) Write the probability that the improvement of the score of the student is greater than 6 marks.

[1]

4. A social work is investigating whether organizing an anti-smoking event in public can decrease the mean smoking rates of citizens. The table below shows the number of times for 6 different people to smoke in a day before the anti-smoking event (x) and after the anti-smoking event (y):

Person	A	B	C	D	E	F
Before anti-smoking event	11	7	13	5	4	8
After anti-smoking event	5	6	10	3	2	8

An appropriate test is conducted at a 5% significance level.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p -value.

[2]

(c) State the conclusion of the test with a reason.

[2]

Two people are randomly selected.

(d) Find the probability that both of them have smaller numbers of times for smoking after the anti-smoking event.

[2]

78 Paper 1 – Binomial and Poisson Variables

Example

A coin is tossed for 500 times. From the observation, 286 results are shown as 'Head's.

A hypothesis test is conducted at a 5% significance level to test whether the coin is biased such that the probability of getting a 'Head' is larger.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

Solution

(a) (i) $H_0: p = 0.5$ A1

(ii) $H_1: p > 0.5$ A1

[2]

(b) $\text{P}(X \geq 286) = 1 - \text{P}(X \leq 285)$ (M1) for valid approach

$\text{P}(X \geq 286) = 0.0007362287541$

Thus, the p-value is 0.000736. A1

[2]

(c) The null hypothesis is rejected. A1

As $p\text{-value} < 0.05$. A1

[2]

Exercise 78

1. A die is tossed for 840 times. From the observation, 99 results are shown as '1's.

A hypothesis test is conducted at a 1% significance level to test whether the die is biased such that the probability of getting a '1' is less than 0.15.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

2. The number of phone calls received by a customer service counter per hour follows a Poisson distribution with mean λ. In a particular hour, there were 13 phone calls.

A hypothesis test is conducted at a 5% significance level to test whether λ is greater than 10.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

3. The number of defective items of a supply chain is studied. A sample of 100 items are selected and n items are found to be defective.

A hypothesis test is conducted at a 5% significance level to test whether the proportion of defective items is greater than 0.03.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

Consider the following table:

	$X \sim \text{B}(100, 0.03)$
$\text{P}(0 \le X \le 4)$	0.8178548035
$\text{P}(X = 5)$	0.101308065
$\text{P}(X = 6)$	0.0496096195
$\text{P}(X = 7)$	0.0206037006
$\text{P}(8 \le X \le 100)$	0.0106238089

(b) (i) Find the least value of n such that the null hypothesis of the test is rejected.

(ii) Hence, write down the greatest value of n such that the null hypothesis of the test is not rejected.

[3]

4. The number of online transactions per day follows a Poisson distribution with mean λ. It is found that there are n online transactions on a particular day.

A hypothesis test is conducted at a 1% significance level to test whether λ is greater than 2.5.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

Consider the following table:

	$X \sim \text{Po}(2.5)$
$\text{P}(0 \le X \le 5)$	0.9579789618
$\text{P}(X = 6)$	0.0278337262
$\text{P}(X = 7)$	0.0099406165
$\text{P}(X = 8)$	0.0031064427
$\text{P}(X \ge 9)$	0.0011402528

(b) (i) Find the least value of n such that the null hypothesis of the test is rejected.

(ii) Hence, write down the greatest value of n such that the null hypothesis of the test is not rejected.

[3]

79 Paper 2 – Type I and Type II Errors

Example

The random variable X follows a Poisson distribution with parameter λ, representing the rate of occurrences of an event per hour.

A hypothesis test is conducted at a particular significance level to test whether λ is less than 5.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that at most one occurrence of the event is observed in a particular hour.

(b) Find the probability that a Type I error is made.

[2]

The actually value of λ is 6.

(c) Find the probability that a Type II error is made.

[2]

Another random variable Y follows a Binomial distribution with parameters n and q, representing the number of trials in an experiment and the probability of success of an event respectively.

A hypothesis test is conducted at a 5% significance level to test whether q is less than 0.25. From a random sample of size 20, there are 3 observed successes.

(d) Write down the critical region for testing q, giving the answer in the form $a \le Y \le b$.

[2]

(e) Find the p-value.

[2]

(f) State the conclusion of the test with a reason.

[2]

Solution

(a) (i) $H_0: \lambda = 5$ A1

(ii) $H_1: \lambda < 5$ A1

[2]

(b) The required probability
$= \mathrm{P}(X \leq 1 \mid \lambda = 5)$ (M1) for valid approach
$= 0.040427682$
$= 0.0404$ A1

[2]

(c) The required probability
$= \mathrm{P}(X \geq 2 \mid \lambda = 6)$ (M1) for valid approach
$= 1 - \mathrm{P}(X \leq 1 \mid \lambda = 6)$
$= 0.9826487348$
$= 0.983$ A1

[2]

(d) $0 \leq Y \leq 1$ A2

[2]

(e) $\mathrm{P}(Y \leq 3) = 0.2251560477$ (M1) for valid approach
Thus, the p-value is 0.225. A1

[2]

(f) The null hypothesis is not rejected. A1
As $p\text{-value} > 0.05$. A1

[2]

Exercise 79

1. The random variable X follows a Poisson distribution with parameter λ, representing the rate of occurrences of an event per day.

A hypothesis test is conducted at a particular significance level to test whether λ is greater than 9.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that at least 14 occurrences of the event is observed in a particular day.

(b) Find the probability that a Type I error is made.

[2]

The actually value of λ is 13.

(c) Find the probability that a Type II error is made.

[2]

Another random variable Y follows a Binomial distribution with parameters n and q, representing the number of trials in an experiment and the probability of success of an event respectively.

A hypothesis test is conducted at a 1% significance level to test whether q is less than 0.12. From a random sample of size 100, there are 6 observed successes.

(d) Write down the critical region for testing q, giving the answer in the form $a \le Y \le b$.

[2]

(e) Find the p-value.

[2]

(f) State the conclusion of the test with a reason.

[2]

2. The random variable X follows a Binomial distribution with parameters 50 and q, representing the number of trials in an experiment and the probability of success of an event respectively.

A hypothesis test is conducted at a particular significance level to test whether q is greater than 0.92.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that at least 49 successes are observed out of 50 trials.

(b) Find the probability that a Type I error is made.

[2]

The actually value of q is 0.94.

(c) Find the probability that a Type II error is made.

[2]

Another random variable Y follows a Poisson distribution with parameter λ, representing the rate of occurrences of an event per day.

A hypothesis test is conducted at a 5% significance level to test whether λ is greater than 3. It is observed that 5 occurrences of the event is observed in a particular day.

(d) Write down the critical region for testing λ, giving the answer in the form $Y \geq a$.

[2]

(e) Find the p-value.

[2]

(f) State the conclusion of the test with a reason.

[2]

3. The random variable X follows a normal distribution with parameter μ, where μ represents the mean value of a population. The population variance is 5.

A hypothesis test is conducted at a particular significance level to test whether μ is greater than 250.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that the mean of a sample of size 25 is greater than 251.

(b) Find the probability that a Type I error is made.

[2]

The actually value of μ is 250.5.

(c) Find the probability that a Type II error is made.

[2]

Another random variable Y is normally distributed with parameter λ, where λ represents the mean value of a new population. The population variance is 11.

A hypothesis test is conducted at a 5% significance level to test whether λ is not equal to 300. It is observed that the mean is 279 in a random sample of size 15.

(d) Write down the critical region for testing λ, giving the answer in the form $(\bar{Y} \leq a \text{ or } \bar{Y} \geq b)$.

[2]

(e) State the conclusion of the test with a reason.

[2]

Assume that the level of significance is changed to 10%.

(f) Write down the new critical region for testing λ, giving the answer in the form $(\bar{Y} \leq a \text{ or } \bar{Y} \geq b)$.

[2]

4. The random variable X follows a normal distribution with parameter μ, where μ represents the mean value of a population. The population variance is 16.

A hypothesis test is conducted at a particular significance level to test whether μ is not equal to 100.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

The null hypothesis is rejected if it is observed that the mean of a sample of size 36 is less than 99 or greater than 101.

(b) Find the probability that a Type I error is made.

[2]

The actually value of μ is 101.

(c) Find the probability that a Type II error is made.

[2]

Another random variable Y is normally distributed with parameter λ, where λ represents the mean value of a new population. The population variance is 20.

A hypothesis test is conducted at a 5% significance level to test whether λ is less than 25. It is observed that the mean is 20 in a random sample of size 60.

(d) Write down the critical region for testing λ, giving the answer in the form $\overline{Y} \le a$.

[2]

(e) State the conclusion of the test with a reason.

[2]

Assume that the level of significance is changed to 1%.

(f) Write down the new critical region for testing λ, giving the answer in the form $\overline{Y} \le a$.

[2]

80 Paper 2 – Population Correlation Coefficient

Example

The following table shows the repair cost $(\$y)$ of a type of machine of a particular age $(x \text{ years})$.

Age (x years)	3	5	7	10	12
Repair cost (\$ y)	500	750	800	1200	1800

It is assumed that the two variables follow a bivariate normal distribution with product moment correlation coefficient ρ. A hypothesis test is conducted at a 5% significance level to test whether there is a positive correlation between the two variables.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

This data can be modelled by the regression line with equation $y = ax + b$.

(d) (i) Write down the value of a and of b.

(ii) Explain what the gradient a represents.

[3]

(e) Use the model to estimate the repair cost of a machine with age 9 years.

[2]

(f) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

Solution

(a) (i) $H_0: \rho = 0$ A1

(ii) $H_1: \rho > 0$ A1

[2]

(b) $p\text{-value} = 0.0055937394$ (A1) for correct value

$p\text{-value} = 0.00559$ A1

[2]

(c) The null hypothesis is rejected. A1

As $p\text{-value} < 0.05$. A1

[2]

(d) (i) $a = 133.0827068$

$a = 133$ A1

$b = 25.18796992$

$b = 25.2$ A1

(ii) a represents the average increase of the repair cost when the age of a machine is increased by 1 year. A1

[3]

(e) The estimated age

$= 133.0827068(9) + 25.18796992$ (A1) for substitution

$= 1222.932331$

$= \$1220$ A1

[2]

(f) (i) $r = 0.9555151843$

$r = 0.956$ A1

(ii) $R^2 = 0.9130092674$

$R^2 = 0.913$ A1

(iii) 91.3% of the variability of the data is explained by the regression model. A1

[3]

Exercise 80

1. The following table shows the quiz score $(y \text{ marks})$ of a student with revision time $(x \text{ hours})$.

Revision time (x hours)	2	4	6	8	10	12
Quiz score (y)	90	80	82	60	55	50

It is assumed that the two variables follow a bivariate normal distribution with product moment correlation coefficient ρ. A hypothesis test is conducted at a 10% significance level to test whether there is a negative correlation between the two variables.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

This data can be modelled by the regression line with equation $y = ax + b$.

(d) (i) Write down the value of a and of b.

(ii) Explain what the intercept b represents.

[3]

(e) Use the model to estimate the quiz score of a student who has revised for 7 hours.

[2]

(f) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

2. The following table shows the number of webpages y visited by a student who spend x minutes to access the internet.

Time (x minutes)	5	10	15	20	25	30
Number of webpages (y)	4	3	1	16	21	23

It is assumed that the two variables follow a bivariate normal distribution with product moment correlation coefficient ρ . A hypothesis test is conducted at a 5% significance level to test whether there is any correlation between the two variables.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p -value.

[2]

(c) State the conclusion of the test with a reason.

[2]

This data can be modelled by the regression line with equation $y = ax + b$.

(d) (i) Write down the value of a and of b .

(ii) Explain what the gradient a represents.

[3]

(e) Use the model to estimate the number of webpages visited by a student who spend 22 minutes to access the internet, giving the answer correct to the nearest integer.

[2]

(f) (i) Write down the correlation coefficient.

(ii) Write down the coefficient of determination.

(iii) Hence, interpret the coefficient of determination.

[3]

3. The following table shows the temperature $(y^\circ\text{C})$ of the motor of a car travelling with speed $(x\ \text{ms}^{-1})$.

Speed $(x\ \text{ms}^{-1})$	30	40	50	60	70
Temperature $(y^\circ\text{C})$	98	122	137	115	100

It is assumed that the two variables follow a bivariate normal distribution with product moment correlation coefficient ρ. A hypothesis test is conducted at a 5% significance level to test whether there is a positive correlation between the two variables.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

This data is suggested to be modelled by the regression line with equation $y = ax + b$.

(d) Is this suggestion a valid approach? Explain your answer.

[2]

(e) (i) Write down the coefficient of determination.

(ii) Hence, interpret the coefficient of determination.

[2]

It is suggested that the relationship between the variables can be modelled by the new regression equation $y = -0.1x^2 + 8x - 70$.

(f) Write down SS_{res}, the sum of square residuals.

[2]

It is also given that the coefficient of determination under the new model is 0.9037.

(g) By using $R^2 = 1 - \frac{SS_{res}}{SS_{tot}}$, find SS_{tot}, the total sum of squares.

[2]

4. The following table shows the amount of calories $(y \text{ kcal})$ of a particular kind of drinks of 250 mL containing sugar of amount $(x \text{ g})$.

Amount of sugar $(x \text{ g})$	3	6	9	12	15
Amount of calories $(y \text{ kcal})$	120	115	112	100	115

It is assumed that the two variables follow a bivariate normal distribution with product moment correlation coefficient ρ. A hypothesis test is conducted at a 10% significance level to test whether there is a negative correlation between the two variables.

(a) (i) Write down the null hypothesis of the test.

(ii) Write down the alternative hypothesis of the test.

[2]

(b) Find the p-value.

[2]

(c) State the conclusion of the test with a reason.

[2]

This data is suggested to be modelled by the regression line with equation $y = ax + b$.

(d) Is this suggestion a valid approach? Explain your answer.

[2]

(e) (i) Write down the coefficient of determination.

(ii) Hence, interpret the coefficient of determination.

[2]

It is suggested that the relationship between the variables can be modelled by the new regression equation $y = 0.25x^2 - 5x + 135$.

(f) Write down the exact value of SS_{res}, the sum of square residuals.

[2]

It is also given that the coefficient of determination under the new model is 0.58.

(g) By using $R^2 = 1 - \dfrac{SS_{res}}{SS_{tot}}$, find SS_{tot}, the total sum of squares.

[2]

Chapter

20

Paper 3 Analysis

SUMMARY POINTs

- ✓ Nature of paper: Structured question
- ✓ Time allowed: 60 minutes
- ✓ Maximum mark: 55 marks
- ✓ Number of questions: 2
- ✓ Mark range per question: 25 marks to 30 marks
- ✓ Weighting: 20% of the total mark

SUMMARY POINTs

✓ Ways of assessing:

1. Identify a hypothesis test
2. State
   - (a) a condition
   - (b) a reason
   - (c) an assumption
3. Write down
   - (a) the value of a quantity
   - (b) the formula of a quantity
4. Find
   - (a) the value of a quantity
   - (b) the formula of a quantity
5. Solve an equation
6. Perform a hypothesis test
7. Show
   - (a) a quantity equals to a value
   - (b) the formula of a quantity
8. Estimate the value of a quantity
9. Predict the value of a quantity
10. Sketch a graph
11. Suggest an improvement
12. Explain a reason
13. Describe a result
14. Verify
    - (a) the value of a quantity
    - (b) the expression of a quantity
15. Comment on
    - (a) the validity of an argument
    - (b) a statement

Solutions of Chapter 20

81 Paper 3 – Practical Example

Example

Rodrigo and Thiago are two full-time football players in Spain. They are belonged to two different teams in a football league. As being the main striker of their teams, their numbers of goals are investigated.

Part I – Models for the Population 1

The following table shows the number of goals x scored by Rodrigo in the football league each year, where n is the number of years after 2011:

n	1	2	3	4	5	6	7	8
x	10	15	20	27	28	24	18	12

(a) The relationship between the variables is modelled by the regression equation $x(n) = an^2 + bn + c$.

(1) Write down the values of a, b and c.

(2) Hence, estimate the number of goals scored by Rodrigo in 2020, giving the answer correct to the nearest integer.

(3) Comment the validity of this estimation.

(b) A hypothesis test is conducted at a 5% significance level to test the probability of a shoot tried by Rodrigo converted to a goal is greater than 0.6.

From 2012 to 2019, Rodrigo has tried 251 shoots in total.

(1) Write down the total number of goals scored by Rodrigo from 2012 to 2019.

(2) Write down the null hypothesis of the test.

(3) Write down the alternative hypothesis of the test.

(4) Find the p-value.

(5) State the conclusion of the test with a reason.

Part I Solution

(a) (1) $a = -1.30952381$, $b = 12.28571429$, $c = -2.642857143$
$a = -1.31$, $b = 12.3$, $c = -2.64$

(2) The estimated number of goals
$= -1.30952381(9)^2 + 12.28571429(9) - 2.642857143$
$= 1.857142857$
$= 2$

(3) This estimation is not valid as this is an extrapolation.

(b) (1) 154

(2) $H_0:\ p = 0.6$

(3) $H_1:\ p > 0.6$

(4) $\mathrm{P}(X \geq 154) = 1 - \mathrm{P}(X \leq 153)$
$\mathrm{P}(X \geq 154) = 0.3557560215$
Thus, the p-value is 0.356.

(5) The null hypothesis is not rejected.
As $p\text{-value} > 0.05$.

The following table shows the number of goals y scored by Thiago in the football league each year, where n is the number of years after 2011:

n	1	2	3	4	5	6	7	8
y	12	8	7	18	23	26	22	17

(c) The relationship between the variables is modelled by the regression equation $y(n) = an^3 + bn^2 + cn + d$.

(1) Write down the equation of the least square regression curve, giving the coefficients correct to 6 significant figures if necessary.

(2) Write down the coefficient of determination.

(3) Hence, interpret the coefficient of determination.

Thiago's personal trainer would like to conduct a χ^2 goodness of fit test at a 10% significance level to determine if the actual number of goals can be modelled by the following distribution:

n	1	2	3	4	5	6	7	8
Expected frequency	14	14	14	14	21	21	21	14

(d) (1) Write down the degree of freedom of the test.

(2) Find the value of χ^2_{calc}, the test statistic.

The critical value is given by 12.017.

(3) State the conclusion of the test with a reason.

Part II Solution

(c) (1) $y(n) = -0.4621212121n^3 + 5.816017316n^2$
$-18.00757576n + 24.21428571$
$y(n) = -0.462121n^3 + 5.81602n^2 - 18.0076n + 24.2143$

(2) $R^2 = 0.9281039957$
$R^2 = 0.928$

(3) 92.8% of the variability of the data is explained by the regression model.

(d) (1) 7

(2) 9.57

(3) The null hypothesis is not rejected.
As $\chi^2_{calc} < 12.017$.

Part III – Combining the Two Populations

Sergio, a football data analyst, is going to investigate the combined data of Rodrigo and Thiago, summarized as follows:

n	1	2	3	4	5	6	7	8
x	10	15	20	27	28	24	18	12
y	12	8	7	18	23	26	22	17

Sergio would like to perform a paired t-test at a 5% significance level to determine whether on average Rodrigo scores more goals than Thiago each year.

(e) (1) Write down the null hypothesis of the test.

(2) Write down the alternative hypothesis of the test.

(3) Find the p-value.

(4) Write down the value of t, the t-statistic.

(5) State the conclusion of the test with a reason.

The product moment correlation coefficient of the numbers of goals scored by the two players is ρ. Sergio would also like to perform a hypothesis test at a 10% significance level to test whether there is a positive correlation between the two variables.

(f) (1) State an assumption for testing ρ.

(2) Write down the null hypothesis of the test.

(3) Write down the alternative hypothesis of the test.

(4) Find the p-value.

(5) State the conclusion of the test with a reason.

Let $s = x + y$ be the sum of the number of goals scored by Rodrigo and Thiago in each year. Let $s(n)$ be the cubic regression equation used to model the relationship between n and s.

(g) (1) Write down the equation of $s(n)$, giving the coefficients correct to 6 significant figures if necessary.

(2) Is it true that $s(n) = x(n) + y(n)$? Explain your answer.

Part III Solution

(e) (1) Let μ_d, $d = x - y$ be the mean of the differences of the numbers of goals scored by Rodrigo and Thiago.

$H_0:\ \mu_d = 0$

(2) $H_1:\ \mu_d > 0$

(3) $p\text{-value} = 0.1534961923$

$p\text{-value} = 0.153$

(4) 1.10

(5) The null hypothesis is not rejected.

As $p\text{-value} > 0.05$.

(f) (1) The numbers of goals scored by the two players follow a bivariate normal distribution.

(2) $H_0:\ \rho = 0$

(3) $H_1:\ \rho > 0$

(4) $p\text{-value} = 0.0918559883$

$p\text{-value} = 0.0919$

(5) The null hypothesis is rejected.

As $p\text{-value} < 0.1$.

(g) (1) $s(n) = -0.5429292929n^3 + 5.597402597n^2$

$-9.883477633n + 25.57142857$

$s(n) = -0.542929n^3 + 5.59740n^2 - 9.88348n + 25.5714$

(2) $x(n) + y(n) = (-1.30952381n^2 + 12.28571429n - 2.642857143)$

$+(-0.4621212121n^3 + 5.816017316n^2 - 18.00757576n + 24.21428571)$

$x(n) + y(n) = -0.4621212121n^3 + 4.506493506n^2$

$-5.72186147n + 21.57142857$

Thus, $s(n) = x(n) + y(n)$ is not true.

Quick Practice

Nina and Olga are two authors selling the books they individually published. The numbers of books sold by them in each year are investigated.

Part I – Models for the Population 1

The following table shows the number of books x, sold by Nina each year, where n is the number of years after 2000:

n	1	2	3	4	5	7	9
x	400	380	365	350	348	357	311

(a) The relationship between the variables is modelled by the regression equation $x(n) = an^3 + bn^2 + cn + d$.

(1) Write down the values of a, b, c and d.

(2) Hence, estimate the number of books sold by Nina in 2006, giving the answer correct to the nearest integer.

(3) Comment the validity of this estimation.

(b) Nina claims that the population mean μ of the number of books sold by her from 2003 to 2012 is less than 370. A one sample t-test is conducted at a 5% significance level to test Nina's claim.

(1) Write down the exact value of the sample mean of the number of books sold from 2003 to 2012.

(2) Write down the null hypothesis of the test.

(3) Write down the alternative hypothesis of the test.

(4) Find the p-value.

(5) State the conclusion of the test with a reason.

Part II – Models for the Population 2

The following table shows the number of books y, sold by Olga each year, where n is the number of years after 2000:

n	1	2	3	4	5	7	9
y	390	386	381	375	368	349	330

(c) The relationship between the variables is modelled by the regression equation $y(n) = a \cdot b^n$.

(1) Write down the equation of the least square regression curve, giving the coefficients correct to 5 significant figures if necessary.

(2) Write down the coefficient of determination.

(3) Hence, interpret the coefficient of determination.

Olga would like to conduct a χ^2 goodness of fit test at a 2.5% significance level to determine if the actual number of books sold by her can be modelled by the following distribution:

n	1	2	3	4	5	7	9
Expected frequency	400	400	380	380	360	340	320

(d) (1) Write down the degree of freedom of the test.

(2) Find the value of χ^2_{calc}, the test statistic.

The critical value is given by 14.449.

(3) State the conclusion of the test with a reason.

Part III – Combining the Two Populations

An investor is going to investigate the data of the numbers of books sold by Nina and Olga, summarized as follows:

n	1	2	3	4	5	7	9
x	400	380	365	350	348	357	311
y	390	386	381	375	368	349	330

The investor would like to perform a suitable statistical test at a 10% significance level to determine whether on average Olga can sell more books than Nina each year.

(e) (1) Write down the null hypothesis of the test.

(2) Write down the alternative hypothesis of the test.

(3) Find the p-value.

(4) State the conclusion of the test with a reason.

The product moment correlation coefficient of the numbers of books sold by the two authors is ρ. The investor would also like to perform a hypothesis test at a 5% significance level to test whether there is any correlation between the two variables.

(f) (1) State an assumption for testing ρ.

(2) Write down the null hypothesis of the test.

(3) Write down the alternative hypothesis of the test.

(4) Find the p-value.

(5) State the conclusion of the test with a reason.

Let $d = y - x$. The following table shows the distribution of d:

n	1	2	3	4	5	7	9
x	400	380	365	350	348	357	311
y	390	386	381	375	368	349	330
d	−10	6	16	25	20	−8	19

(g) (1) Construct a 90% confidence interval for the population mean of d.

(2) Is the above result consistent with the conclusion of the hypothesis test in (e)? Explain your answer.

Exercise 81

1. This question aims at investigate the relationship between the second-order differential equation and the eigenvalues of the corresponding coupled system of first-order differential equations.

Consider the coupled system $\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}x} = 4v + 12y \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = v \end{cases}$.

(a) Show that the coupled system can be expressed as $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\dfrac{\mathrm{d}y}{\mathrm{d}x} - 12y = 0$.

[2]

(b) The coupled system can be expressed by a matrix equation $\dot{\mathbf{Y}} = \mathbf{MY}$, where $\mathbf{M}$ is a 2×2 matrix, and $\dot{\mathbf{Y}} = \begin{pmatrix} \dfrac{\mathrm{d}v}{\mathrm{d}x} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} v \\ y \end{pmatrix}$ are two 2×1 matrices.

(1) Write down $\mathbf{M}$.

Let λ_1 and λ_2 be the eigenvalues of $\mathbf{M}$, where $\lambda_1 < \lambda_2$.

(2) Find the characteristic polynomial of $\mathbf{M}$.

(3) Hence, write down the values of λ_1 and λ_2.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the eigenvectors of $\mathbf{M}$ corresponding to λ_1 and λ_2 respectively.

(4) Write down $\mathbf{v}_1$ and $\mathbf{v}_2$.

(5) Hence, write down the general solution for y.

(6) By considering the expressions of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, show that the general solution for y satisfies the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\dfrac{\mathrm{d}y}{\mathrm{d}x} - 12y = 0$.

[12]

(c) (1) Suggest a relationship between the distinct real eigenvalues of $\mathbf{M}$ and the coefficients of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y in $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} - 12y = 0$.

(2) Hence, write down the corresponding second-order differential equation for a general coupled system $\dot{\mathbf{Y}} = \mathbf{NY}$, where $\mathbf{N}$ is a 2×2 matrix, $\dot{\mathbf{Y}} = \begin{pmatrix} \frac{\mathrm{d}v}{\mathrm{d}x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} \end{pmatrix}$, $\mathbf{Y} = \begin{pmatrix} v \\ y \end{pmatrix}$, and the characteristic polynomial of $\mathbf{N}$ is given by $\lambda^2 + b\lambda + c = 0$ with distinct real roots.

[4]

(d) Using the above results, for the second-order differential equation $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 12\frac{\mathrm{d}y}{\mathrm{d}x} + 27y = 0$ for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 15$ and $y = 3$ when $x = 0$, find

(1) the general solution ;

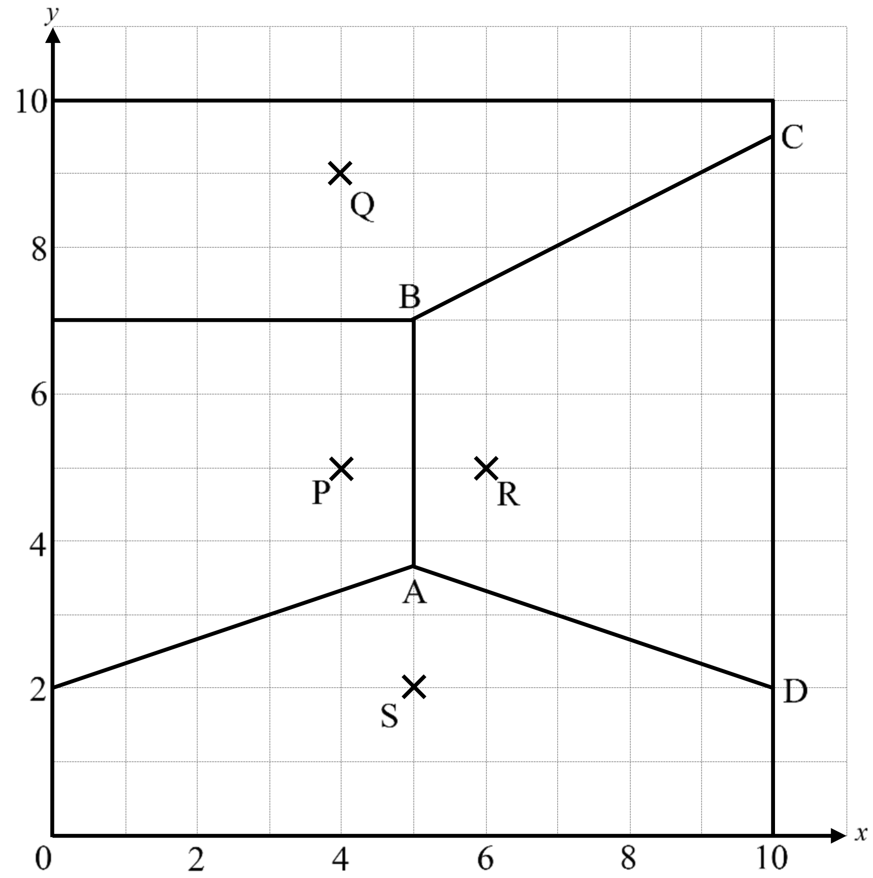
(2) the particular solution.

[8]

20

2. This question aims at investigate an urban planning problem involving Voronoi diagrams and graph theory.

There are four housing estates P, Q, R and S in a town. The diagram below shows the Voronoi diagram of the four housing estates, where 1 unit represents 100 m. The points $\mathrm{A}\left(5, \frac{11}{3}\right)$ and B(5, 7) are the intersections of the boundaries of Voronoi cells. BC and AD intersect the line $x = 10$ at C and D respectively.



The town can be modelled by the region in the coordinate plane where $0 \le x \le 10$ and $0 \le y \le 10$.

In order to minimize the noise pollution level due to traffic, the urban planner of the town decides to build highways at the location which are as far as possible from the housing estates, covering all edges of the Voronoi diagram and the external boundaries of the town.

(a) (1) Write down the gradient of QR .

(2) Hence, write down the gradient of the perpendicular bisector of QR .

(3) Find the equation of the perpendicular bisector of QR .

(4) Find the coordinates of C .

(5) Hence, find the length of the highway connecting B and C .

[9]

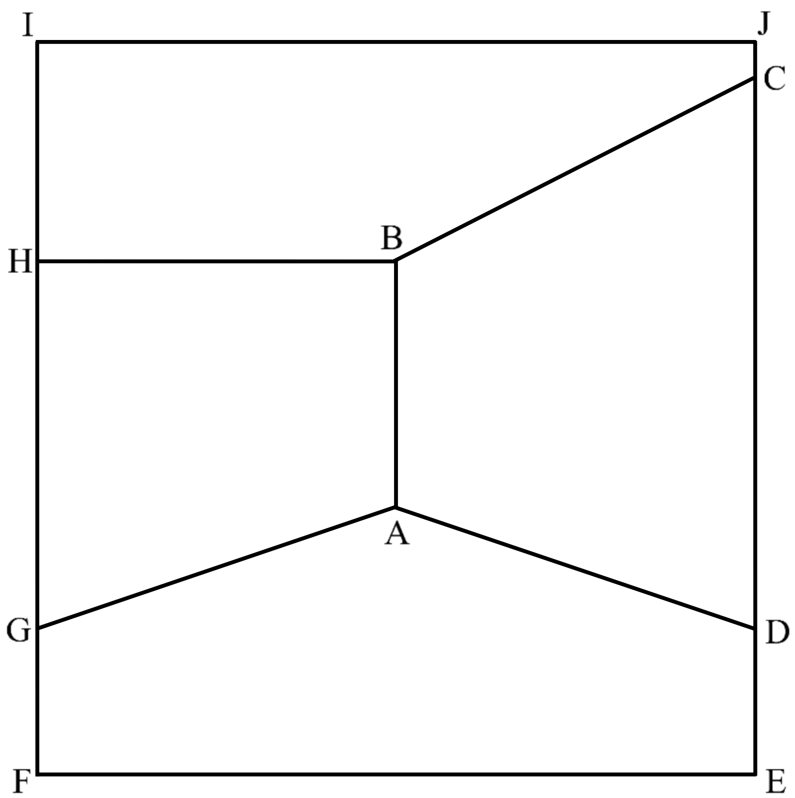
It is given that the length of the highway connecting A and D is 527 m, correct to the nearest metre.

(b) (1) Write down the length of the highway connecting B and A .

(2) Using the results from (a)(5) and (b)(1), find the total length of the highway in the town.

[4]

In order to reduce the travelling time of the public transport, the urban planner is going to introduce the use of roundabouts to replace the use of traffic lights. The roundabouts are at all junctions A , B , C , D , G and H where more than one highway intersect, and the four vertices E , F , I and J , as shown in the following graph:



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(c) Write down the number of vertices of

(1) odd degree;

(2) even degree.

[2]

(d) Is Eulerian circuit exists in the above graph? Explain your answer.

[2]

Due to the difference between the populations of the four housing estates, the travelling time along the highway may not be proportional to the length of the highway. The following table shows the average times, in seconds, need for a bus to travel along each highway:

Highway	AB	AD	AG	BC	BH	CD	CJ
Time needed	30	40	60	35	30	45	10
Highway	DE	EF	FG	GH	HI	IJ	
Time needed	25	65	25	50	20	55	

Kruskal's algorithm is used to find the minimum spanning tree for this graph.

(e) (1) State the highway with the shortest time needed to travel along.

(2) By using the algorithm, find the minimum spanning tree.

(3) Write down the corresponding total time of the minimum spanning tree.

[5]

(f) Use the Chinese postman algorithm to find a possible route of minimum time needed for a bus to travel along all highways, starting and finishing at E.

[4]

By adding three highways in the town, an Eulerian circuit will exist.

(g) Suggest one possible combination of three highways to be added in the town.

[3]

Answers

Chapter 1

Exercise 1

1.1 **(a)** $f^{-1}(x)=\frac{x+1}{8}$

(b) **(i)** 20
(ii) 159

1.2 **(a)** $f^{-1}(x)=\frac{x+3}{2}$

(b) **(i)** -7
(ii) 4

1.3 **(a)** 12
(b) **(i)** 96
(ii) 10

1.4 **(a)** 5
(b) $\sqrt{3a}$

Exercise 2

2.1 **(a)** $\{y : y \le 10\}$
(b) -7
(c) $f^{-1}(x)=\sqrt{10-x}-7$

2.2 **(a)** $f(x)=(x-4)^2+24$
(b) 4
(c) $f^{-1}(x)=\sqrt{x-24}+4$

2.3 **(a)** $h=13$
(b) -13
(c) $f^{-1}(x)=\sqrt{\frac{x}{4}}-13$

2.4 **(a)** (1, 0)
(b) 1
(c) $f^{-1}(x)=\sqrt{\sqrt{x}+4}+3$

Exercise 3

3.1 **(a)** $g(x)=(x-3)^2+1$
(b) **(i)** $p=-2$
(ii) $q=-6$

3.2 **(a)** $g(x)=(x+8)^2-6$
(b) **(i)** $p=0$
(ii) $q=206$

3.3 **(a)** $g(x)=-x^2+1$
(b) $r=3$

3.4 **(a)** $g(x)=5x^2+20x+30$
(b) $(-2, 10)$
(c) **(i)** $p=3$
(ii) $q=-12$

Exercise 4

4.1 **(a)** $f^{-1}(-2)=-3$
(b) $(f\circ f)(5)=2$
(c) Refer to solution

4.2 **(a)** $f^{-1}(2)=0$
(b) $(f\circ f)(4)=2$
(c) Refer to solution

4.3 **(a)** $-2\le x\le 3$
(b) $(f^{-1}\circ f^{-1})(1)=3$
(c) Refer to solution

4.4 **(a)** $-4\le x\le 3$
(b) $(f^{-1}\circ f^{-1})(3)=4$
(c) Refer to solution

Exercise 5

5.1 **(a)** $C=12.5s$
(b) \$325
(c) $=\$393$
(d) $C=0.4s^{\frac{1}{3}}$

5.2 **(a)** $P = \frac{180}{A}$

(b) $2.4

(c) The price of a tetrahedron model of a large surface area will approach $0.

(d) -1

5.3 **(a)** $W = 18\sqrt{l}$

(b) 441 g

(c) Translate to the left by 3 units, followed by a vertical stretch of scale factor 2

5.4 **(a)** $f(x) = \frac{6}{x^2}$

(b) $x = 0$

(c) $g(x) = \frac{24}{(x-3)^2}$

Chapter 2

Exercise 6

6.1 **(a)** $\log\frac{1}{25} + \log\frac{1}{40} = -3$

(b) $\ln e^{7.5} - \ln e\sqrt{e} = 6$

6.2 **(a)** $\log 0.8 + \log 1250 = 3$

(b) $\ln\sqrt[3]{e} - \ln e^{\frac{4}{3}} = -1$

6.3 **(a)** $\log 112 + \log\frac{25}{4} - \log 7 = 2$

(b) $x = 9$

6.4 **(a)** $\ln\frac{1}{3} + \ln 45 - \ln 15 = 0$

(b) $x = 11$

Exercise 7

7.1 **(a)** $f^{-1}(x) = 10^{3x}$

(b) $\{y : y > 0\}$

(c) $(f^{-1} \circ g)(\sqrt{10}) = 1000000$

7.2 **(a)** $f^{-1}(x) = 0.25\ln x$

(b) $\{x : x > 0\}$

(c) $(g \circ f^{-1})(16) = 1$

7.3 **(a)** $f^{-1}(x) = e^{x-3}$

(b) $\{y : y > 0\}$

(c) $(f \circ g)(2) = 0$

7.4 **(a)** $f^{-1}(x) = \frac{1}{3}\log x$

(b) $\{y : y \in \mathbb{R}\}$

(c) $(g \circ f)(x) = 9x^2 + 6x + 1$

Exercise 8

8.1 **(a)** **(i)** 2 million

(ii) 5.85 million

(b) $\{B : 2 \le B < 12\}$

8.2 **(a)** 3 g

(b) 30th July

(c) $W = 27$

8.3 **(a)** $C = 5$

(b) $t = 24.6$

(c) $P = 600000$

8.4 **(a)** $879000

(b) $t = 15.9$

(c) The value of the vase will approach $5000000 after a long period of time.

Exercise 9

9.1 **(a)** $\ln T = (\ln k)x + \ln h$

(b) **(i)** $h = 0.1827$

(ii) $k = 1.0618$

9.2 **(a)** $\ln W = k\ln x + \ln h$

(b) $k = -5$

(c) $h = 4.4817$

9.3 **(a)** $b = 18$

(b) $V = e^{3x+18}$

(c) 4.48

9.4 (a) (i) $a = -0.4$

(ii) $b = 2$

(b) $N = e^2 t^{-0.4}$

Exercise 10

10.1 (a) 2500

(b) 5290

(c) 15.5 years

(d) 1470

(e) 12

10.2 (a) 420

(b) 971

(c) 4.15 years

(d) 0.111

(e) 44

10.3 (a) 1050

(b) 37300

(c) 6.21 weeks

(d) 1.19

(e) 10

10.4 (a) 146

(b) 5.78 minutes

(c) (i) 3.5

(ii) 0.130

(d) 32

Chapter 3

Exercise 11

11.1 (a) $r = 0.6$

(b) $S_{10} = -2240$

(c) $S_\infty = -2250$

11.2 (a) $r = \frac{1}{2}$

(b) $u_6 = \frac{3}{2}\ln x$

(c) $S_\infty = 96\ln x$

11.3 (a) $r = e^{-4x}$

(b) $u_7 = e^{-12x}$

(c) $x = 6$

11.4 (a) $r = 3^{-x}$

(b) $u_n = 3^{(11-n)x}$

(c) $S_\infty = \frac{1}{2} \times 3^{11}$

Exercise 12

12.1 (a) $r = \frac{1}{2}$ leads to a finite sum as $-1 < \frac{1}{2} < 1$.

(b) $S_\infty = 40$

12.2 (a) $r = -\frac{1}{3}$ leads to a finite sum as $-1 < -\frac{1}{3} < 1$.

(b) $S_4 = 120$

12.3 (a) $S_\infty = \log_2 x$

(b) $S_6 = \frac{63}{16}$

12.4 (a) $m = -29$

(b) $S_\infty = \frac{243}{4}$

Chapter 4

Exercise 13

13.1 (a) $\cos\theta = -\frac{12}{13}$

(b) $\tan\theta = -\frac{5}{12}$

13.2 (a) $\cos\theta = \frac{5}{6}$

(b) $\sin\theta = -0.553$

13.3 **(a)** **(i)** 0.64
(ii) $\sin\theta = -0.6$
(b) $\tan\theta = 0.75$

13.4 **(a)** **(i)** $-\frac{9}{40}$
(ii) $\sin\theta = \frac{9}{41}$
(b) $\cos\theta = -\frac{40}{41}$

Exercise 14

14.1 **(a)** $x = 0$, $x = 6.28$ or $x = 12.6$
(b) 4

14.2 **(a)** $x = 0.524$ or $x = 2.62$
(b) 1

14.3 **(a)** $x = 3.14$, $x = 6.28$ or $x = 9.42$
(b) 7

14.4 **(a)** $x = 0.262$, $x = 0.785$ or $x = 1.31$
(b) 4

Exercise 15

15.1 **(a)** $h(x) = 3\cos\left(\frac{1}{2}x + \frac{3}{4}\right) - 5$
(b) 4π
(c) $\{y : -8 \le y \le -2\}$

15.2 **(a)** $h(x) = 4\sin\left(4x + \frac{7}{2}\right) - 3$
(b) $\frac{\pi}{2}$
(c) 4

15.3 **(a)** $h(x) = 6\sin\frac{1}{3}x + \frac{37}{2}$
(b) 6
(c) $\left\{y : \frac{25}{2} \le y \le \frac{49}{2}\right\}$

15.4 **(a)** $h(x) = -12\cos\pi x - 1$
(b) 12
(c) $\{y : -13 \le y \le 11\}$

Chapter 5

Exercise 16

16.1 **(a)** 46.2 cm
(b) 110 cm
(c) 742 cm^2

16.2 **(a)** 44.2 cm
(b) 68.2 cm
(c) 265 cm^2

16.3 **(a)** OC = 25.0 cm
(b) **(i)** 4.35 rad
(ii) 1360 cm^2

16.4 **(a)** $\theta = 0.96$ rad
(b) **(i)** 5.32 rad
(ii) 53.9 cm^2

Exercise 17

17.1 **(a)** 43.38 cm
(b) AB = 33.6 cm
(c) 77.0 cm

17.2 **(a)** 1.36 cm^2
(b) 1.26 cm^2
(c) 0.102 cm^2

17.3 **(a)** $A\hat{O}B = 1.86$ rad
(b) 226 cm^2
(c) 109 cm^2

17.4 **(a)** $A\hat{O}B = 1.76$ rad
(b) 42.2 cm
(c) 146 cm

Exercise 18

18.1 **(a)** $A\hat{B}C = 65.8^\circ$ or 114°
(b) $A\hat{C}B = 45.8^\circ$

18.2 **(a)** $B\hat{A}C = 62.4^\circ$ or 118°
(b) $AB = 22.5$ or 11.1
(c) 11.4

18.3 **(a)** $A\hat{B}C = 42.6^\circ$ or 137°
(b) 6.96

18.4 **(a)** $B\hat{A}C = 64.0^\circ$ or 116°
(b) **(i)** $AB < 20$
(ii) 80.1

Chapter 6

Exercise 19

19.1 **(a)** $\frac{1}{3-4i} = \frac{3}{25} + \frac{4}{25}i$
(b) $z^2 = -\frac{7}{625} + \frac{24}{625}i$
(c) $-\frac{7}{625}$

19.2 **(a)** $z = 1 - 2i$
(b) -2

19.3 **(a)** $z = 3 + 4i$
(b) $z^4 = -527 - 336i$
(c) -527

19.4 **(a)** $z = -323 - 36i$
(b) $(i^3 z)^2 = -103033 - 23256i$
(c) -23256

Exercise 20

20.1 **(a)** $z = \frac{3}{5} - \frac{4}{5}i$
(b) 1
(c) -0.927 rad

20.2 **(a)** $z^2 = -\frac{476}{169} - \frac{480}{169}i$
(b) 4
(c) -2.35 rad

20.3 **(a)** **(i)** $z^3 = -\frac{352}{125} - \frac{936}{125}i$
(ii) $(z^3)^* + \frac{352}{125} = \frac{936}{125}i$
(b) $\frac{936}{125}$
(c) $\frac{\pi}{2}$ rad

20.4 **(a)** $|z_1| = 2$
(b) $a = 3$
(c) $\arg(z_2{}^2) = -\frac{2\pi}{3}$ rad

Exercise 21

21.1 **(a)** **(i)** 5
(ii) -12
(b) 13
(c) 338

21.2 **(a)** **(i)** $2 + 10i$
(ii) $-8 + (10 - 10\sqrt{3})i$
(b) 173

21.3 **(a)** Refer to solution
(b) $\arg(\omega) = \frac{3\pi}{4}$ rad
(c) 36

21.4 **(a)** Refer to solution
(b) $\arg(\omega) = \frac{5\pi}{4}$ rad
(c) 202.5

Exercise 22

22.1 **(a)** **(i)** $\frac{z_1}{z_2}=3\text{cis}\frac{2\pi}{3}$

(ii) $\frac{z_1}{z_2}=3e^{\frac{2\pi}{3}i}$

(b) $-\frac{3}{2}$

22.2 **(a)** **(i)** $z_1z_2=2\sqrt{3}\text{cis}\frac{\pi}{6}$

(ii) $z_1z_2=2\sqrt{3}e^{\frac{\pi}{6}i}$

(b) 3

22.3 **(a)** **(i)** $z_1{}^2=4\text{cis}\frac{\pi}{6}$

(ii) 2

(b) **(i)** $z_1{}^2z_2=12\text{cis}\frac{5\pi}{12}$

(ii) $z_1{}^2z_2=12e^{\frac{5\pi}{12}i}$

22.4 **(a)** **(i)** $\frac{1}{81}\text{cis}\frac{2\pi}{3}$

(ii) $-\frac{1}{162}$

(b) **(i)** $\frac{z_2}{z_1{}^4}=9\text{cis}\frac{\pi}{4}$

(ii) $\frac{z_2}{z_1{}^4}=9e^{\frac{\pi}{4}i}$

Exercise 23

23.1 **(a)** The range of $f(x)$ is $\{y:y\le -25\}$, means the graph of $f(x)$ does not have any x-intercept.

(b) $x=2\pm 5i$

(c) 10

23.2 **(a)** The graph of $f(x)$ opens upward and the vertex is above the x-axis, means the graph of $f(x)$ does not have any x-intercept.

(b) $x=-5\pm 8i$

(c) 16

23.3 **(a)** **(i)** 8

(ii) 185

(b) $f(x)=a(x^2-8x+185)$

(c) $a=1$

23.4 **(a)** **(i)** -1

(ii) $\frac{17}{4}$

(b) $f(x)=a\left(x^2+x+\frac{17}{4}\right)$

(c) $a=-4$

Exercise 24

24.1 **(a)** **(i)** 2

(ii) 3

(b) $\frac{\pi}{3}$ s

(c) $z+w=2e^{-0.1i}+3e^{0.25i}$

(d) **(i)** $z=2\text{cis}(-0.1)$

(ii) $w=3\text{cis}0.25$

(e) **(i)** $L=4.93$

(ii) $\alpha=0.110$

(f) $S_1+S_2=4.93\sin(6t+0.110)$

(g) -4.93 mm

24.2 **(a)** **(i)** 5

(ii) 7

(b) 2 s

(c) $z+w=5e^{-0.9i}+7e^{-1.3i}$

(d) **(i)** $z=5\text{cis}(-0.9)$

(ii) $w=7\text{cis}(-1.3)$

(e) **(i)** $L=11.8$

(ii) $\alpha=-1.13$

(f) $W_1+W_2=11.8\cos(\pi t-1.13)$

(g) 1.86

24.3 **(a)** **(i)** 10

(ii) $\frac{\pi}{5}$ s

(b) $z - w = 10e^{0.15i} - 8e^{0.05i}$

(c) **(i)** $z = 10\text{cis}0.15$

(ii) $w = 8\text{cis}0.05$

(d) **(i)** $L = 2.17$

(ii) $\alpha = 0.523$

(e) $S_2 = 2.17\cos(10t - 0.523)$

(f) 9.56

24.4 **(a)** **(i)** 7

(ii) 1 s

(b) $z - w = 6.3e^{-0.5i} - 7e^{-0.95i}$

(c) **(i)** $z = 6.3\text{cis}(-0.5)$

(ii) $w = 7\text{cis}(-0.95)$

(d) **(i)** $L = 3.04$

(ii) $\alpha = 1.07$

(e) $V_2 = 3.04\sin(2\pi t - 1.07)$

(f) $0.5 \le t < 0.557$ or $1.28 < t \le 1.5$

Chapter 7

Exercise 25

25.1 **(a)** 15

(b) $\det \mathbf{B} = 2x^3 - 4x$

(c) $x = 0$

25.2 **(a)** 3

(b) $\det \mathbf{B} = 1 + x^2$

(c) $x = 1$ or $x = 4$

25.3 **(a)** $\det \mathbf{A} = 4e^{2x} - 3e^{x}$

(b) $x = 0$

25.4 **(a)** $\det \mathbf{A} = (\ln x)^2 + 6$

(b) $x = e^2$ or $x = e^3$

Exercise 26

26.1 **(a)** $\mathbf{A}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{7}{3} & -\frac{11}{3} \\ -2 & -5 & -7 \end{pmatrix}$

(b) $\mathbf{B} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{7}{3} \\ -\frac{14}{3} & -\frac{13}{3} & -\frac{40}{3} \\ -9 & -9 & -24 \end{pmatrix}$

26.2 **(a)** $\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$

(b) $\mathbf{B} = \begin{pmatrix} -\frac{33}{5} & \frac{13}{5} & \frac{8}{5} \\ \frac{34}{5} & -\frac{14}{5} & -\frac{9}{5} \\ \frac{42}{5} & -\frac{7}{5} & \frac{28}{5} \end{pmatrix}$

26.3 **(a)** $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{5}{2} & 1 & \frac{9}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{9}{4} & -1 & -\frac{15}{4} \end{pmatrix}$

(b) $\mathbf{C} = \begin{pmatrix} \frac{1}{4} & -\frac{11}{2} & -\frac{7}{4} \\ -\frac{113}{4} & \frac{21}{2} & \frac{177}{4} \\ \frac{15}{4} & -\frac{9}{2} & -\frac{25}{4} \end{pmatrix}$

26.4 **(a)** $\mathbf{A}^{-1}=\begin{pmatrix} \frac{29}{2} & -\frac{53}{2} & 1 \\ \frac{3}{2} & -\frac{5}{2} & 0 \\ -1 & 2 & 0 \end{pmatrix}$

(b)

$\mathbf{C}=\begin{pmatrix} -956 & 4131 & -6559.5 \\ -92 & 403 & -624.5 \\ 71 & -303 & 492 \end{pmatrix}$

Exercise 27

27.1 **(a)** $\mathbf{A}^{-1}=\begin{pmatrix} \frac{6}{13} & \frac{8}{39} & -\frac{29}{39} \\ -\frac{5}{13} & \frac{2}{39} & \frac{22}{39} \\ \frac{1}{13} & -\frac{1}{13} & \frac{2}{13} \end{pmatrix}$

(b) $\mathbf{X}=\begin{pmatrix} -\frac{170}{39} \\ \frac{133}{39} \\ \frac{5}{13} \end{pmatrix}$

27.2 **(a)** $\mathbf{A}^{-1}=\begin{pmatrix} -\frac{14}{37} & \frac{5}{37} & \frac{10}{37} \\ -\frac{45}{37} & \frac{24}{37} & \frac{11}{37} \\ -\frac{13}{37} & \frac{2}{37} & \frac{4}{37} \end{pmatrix}$

(b) $x=-320$, $y=-648$, $z=-276$

27.3 **(a)** $\therefore a=2$

(b) $\mathbf{X}=\begin{pmatrix} -6 \\ 15 \\ -12 \end{pmatrix}$

27.4 **(a)** $p=8$, $q=-24$

(b) $\mathbf{X}=\begin{pmatrix} \frac{1}{16} \\ -\frac{15}{64} \\ \frac{9}{64} \end{pmatrix}$

Exercise 28

28.1 **(a)** $\mathbf{ST}=\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Horizontal stretch with scale factor 6

(c) $(18,-5)$

28.2 **(a)** $\mathbf{ST}=\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) Reflection about the line $x=0$

(c) $(-2,1)$

28.3 **(a)** Rotation clockwise of $\frac{\pi}{6}$ radians about the origin

(b) $(5.46, 1.46)$

(c) 12

28.4 **(a)** Rotation anticlockwise of $\frac{2\pi}{3}$ radians about the origin

(b) $(4, 0)$

(c) 3

Exercise 29

29.1 **(a)** $\lambda^2 - 1$

(b) $\lambda_1 = -1,\ \lambda_2 = 1$

(c) $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(d) $\alpha = 1$

(e) **(i)** $\begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} (-1)^n & 0 \\ 0 & 1 \end{pmatrix}$

(f)

$$\mathbf{A}^n = \begin{pmatrix} \frac{3}{2}(-1)^n - \frac{1}{2} & \frac{3}{2}(-1)^n - \frac{3}{2} \\ -\frac{1}{2}(-1)^n + \frac{1}{2} & -\frac{1}{2}(-1)^n + \frac{3}{2} \end{pmatrix}$$

29.2 **(a)** $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - 12\lambda + 35$

(b) $\lambda_1 = 5,\ \lambda_2 = 7$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(d) $\alpha = -3$

(e) **(i)** $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} 5^n & 0 \\ 0 & 7^n \end{pmatrix}$

(f) $\begin{pmatrix} 555184873 & -545419248 \\ 272709624 & -262943999 \end{pmatrix}$

29.3 **(a)** $\lambda^2 - \lambda + \frac{3}{16}$

(b) $\lambda_1 = \frac{1}{4},\ \lambda_2 = \frac{3}{4}$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 20 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 28 \end{pmatrix}$

(d) **(i)** $\begin{pmatrix} 1 & 1 \\ 20 & 28 \end{pmatrix}$

(ii) $\begin{pmatrix} \left(\frac{1}{4}\right)^n & 0 \\ 0 & \left(\frac{3}{4}\right)^n \end{pmatrix}$

(e)

$$\mathbf{M}^n = \begin{pmatrix} \frac{7}{2}\left(\frac{1}{4}\right)^n - \frac{5}{2}\left(\frac{3}{4}\right)^n & -\frac{1}{8}\left(\frac{1}{4}\right)^n + \frac{1}{8}\left(\frac{3}{4}\right)^n \\ 70\left(\frac{1}{4}\right)^n - 70\left(\frac{3}{4}\right)^n & -\frac{5}{2}\left(\frac{1}{4}\right)^n + \frac{7}{2}\left(\frac{3}{4}\right)^n \end{pmatrix}$$

(f) $\lim_{n\to\infty} f(n) = 0$

29.4 **(a)** $\lambda^2 - \frac{3}{2}\lambda + \frac{1}{2}$

(b) $\lambda_1 = \frac{1}{2},\ \lambda_2 = 1$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(d) **(i)** $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} \left(\frac{1}{2}\right)^n & 0 \\ 0 & 1 \end{pmatrix}$

(e) $\begin{pmatrix} -\left(\frac{1}{2}\right)^n + 2 & \left(\frac{1}{2}\right)^n - 1 \\ -2\left(\frac{1}{2}\right)^n + 2 & 2\left(\frac{1}{2}\right)^n - 1 \end{pmatrix}$

(f) $\lim_{n\to\infty} g(n) = -1$

Exercise 30

30.1 **(a)** **(i)** Refer to solution

(ii) $400a+20b+c=-900$

$25a+5b+c=-1125$

(b) **(i)** $\mathbf{A}=\begin{pmatrix}2500 & 50 & 1\\ 400 & 20 & 1\\ 25 & 5 & 1\end{pmatrix}$

(ii) $\mathbf{B}=\begin{pmatrix}3600\\ -900\\ -1125\end{pmatrix}$

(iii)

$$\mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{1350} & -\frac{1}{450} & \frac{1}{675}\\ -\frac{1}{54} & \frac{11}{90} & -\frac{14}{135}\\ \frac{2}{27} & -\frac{5}{9} & \frac{40}{27}\end{pmatrix}$$

(c) $a=3$, $b=-60$ and $c=-900$

(d) **(i)** $x=-10$ or $x=30$

(ii) -1200

30.2 **(a)** **(i)** Refer to solution

(ii) Refer to solution

(iii) $36a+6b+c=-356$

$100a+10b+c=-516$

(b) **(i)** $\mathbf{A}=\begin{pmatrix}4 & 2 & 1\\ 36 & 6 & 1\\ 100 & 10 & 1\end{pmatrix}$

(ii) $\mathbf{B}=\begin{pmatrix}-228\\ -356\\ -516\end{pmatrix}$

(iii) $\begin{pmatrix}\frac{1}{32} & -\frac{1}{16} & \frac{1}{32}\\ -\frac{1}{2} & \frac{3}{4} & -\frac{1}{4}\\ \frac{15}{8} & -\frac{5}{4} & \frac{3}{8}\end{pmatrix}$

(c) $a=-1$, $b=-24$ and $c=-176$

(d) **(i)** $x=-12$, $x=-8$, $x=-4$

(ii) $y=-384$

30.3 **(a)** **(i)** $\mathbf{M}^2=\begin{pmatrix}1 & 0\\ 10 & 1\end{pmatrix}$

(ii) $\mathbf{M}^3=\begin{pmatrix}1 & 0\\ 15 & 1\end{pmatrix}$

(iii) $\mathbf{M}^{50}=\begin{pmatrix}1 & 0\\ 250 & 1\end{pmatrix}$

(b) **(i)** $s(2)=\begin{pmatrix}2 & 0\\ 15 & 2\end{pmatrix}$

(ii) $s(3)=\begin{pmatrix}3 & 0\\ 30 & 3\end{pmatrix}$

(iii) $s(50)=\begin{pmatrix}50 & 0\\ 6375 & 50\end{pmatrix}$

(c) $p(50)=\begin{pmatrix}1 & 0\\ 6375 & 1\end{pmatrix}$

30.4 **(a)** **(i)** $\mathbf{A}^2 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$

(ii) $\mathbf{A}^3 = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix}$

(iii) $\mathbf{A}^{30} = \begin{pmatrix} 1 & 120 \\ 0 & 1 \end{pmatrix}$

(b) **(i)** $\mathbf{B}^2 = \begin{pmatrix} 1 & 21 \\ 0 & 4 \end{pmatrix}$

(ii) $\mathbf{B}^3 = \begin{pmatrix} 1 & 49 \\ 0 & 8 \end{pmatrix}$

(iii) $\begin{pmatrix} 1 & 7516192761 \\ 0 & 1073741824 \end{pmatrix}$

(c) Refer to solution

Chapter 8

Exercise 31

31.1 $k = -14$ or $k = 3$

31.2 $k \neq 0$ and $k \neq \frac{1}{4}$

31.3 **(a)** $s = -1$

(b) $(3, 2, 1)$

31.4 **(a)** $s = -2$

(b) $k = 11$

Exercise 32

32.1 **(a)** $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$

(b) **(i)** $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -5 \\ -5 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 2 \\ -5 \\ -5 \end{pmatrix}$

32.2 **(a)** **(i)** $\overrightarrow{AB} = \begin{pmatrix} -3 \\ -12 \\ -2 \end{pmatrix}$

(ii) $\mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} -3 \\ -12 \\ -2 \end{pmatrix}$

(b) $k = -\frac{100}{3}$

32.3 **(a)** $\mathbf{v} = \begin{pmatrix} 10 \\ -5 \\ -15 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 10 \\ -5 \\ -15 \end{pmatrix}$

32.4 **(a)** **(i)** $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

(ii) $\overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix}$

(b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -5 \\ 7 \\ -8 \end{pmatrix}$

(c) $\mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} + t\begin{pmatrix} -5 \\ 7 \\ -8 \end{pmatrix}$

Exercise 33

33.1 **(a)** $\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} 5 \\ 12 \\ 0 \end{pmatrix}$

(b) $\frac{36}{13}$

33.2 **(a)** $3\mathbf{a}-2\mathbf{b}=\begin{pmatrix}1\\8\\19\end{pmatrix}$

(b) 5.90

33.3 **(a)** 4.46

(b) **(i)** $|\overrightarrow{OA}|=7$

(ii) $A\hat{O}B=50.5^\circ$

33.4 **(a)** 9.09

(b) **(i)** $|\overrightarrow{OA}|=13$

(ii) $O\hat{A}B=45.7^\circ$

Exercise 34

34.1 **(a)** $\mathbf{r}=\begin{pmatrix}10\\0\end{pmatrix}+t\begin{pmatrix}1\\2\end{pmatrix}$

(b) $(15,10)$

(c) **(i)** $(-13,-4)$

(ii) $(-45,-20)$

(d) $(-25,0)$

(e) $\mathbf{r}=\begin{pmatrix}-25\\0\end{pmatrix}+t\begin{pmatrix}-1\\-1\end{pmatrix}$

(f) 25.3

34.2 **(a)** 13

(b) $\mathbf{r}=\begin{pmatrix}3\\4\end{pmatrix}+t\begin{pmatrix}12\\5\end{pmatrix}$

(c) $\begin{pmatrix}0\\6\end{pmatrix}$

(d) $x=2$

(e) 135°

(f) 3.5 s

34.3 **(a)** $\begin{pmatrix}400\\-70\\-4\end{pmatrix}\text{kmh}^{-1}$

(b) 406 kmh^{-1}

(c) $\mathbf{r}=\begin{pmatrix}-800\\160\\8\end{pmatrix}+t\begin{pmatrix}400\\-70\\-4\end{pmatrix}$

(d) 11:00

(e) $(20,0,0)$

(f) 10:24

34.4 **(a)** $\mathbf{r}=\begin{pmatrix}20\\-10\\0\end{pmatrix}+t\begin{pmatrix}5\\5\\0\end{pmatrix}$

(b) $p=4$

(c) $\begin{pmatrix}5\\-5\\-5\end{pmatrix}\text{s}^{-1}$

(d) $\mathbf{r}=\begin{pmatrix}20\\20\\20\end{pmatrix}+t\begin{pmatrix}5\\-5\\-5\end{pmatrix}$

(e) 4.47

(f) 3.20 seconds after the start of the game

Exercise 35

35.1 **(a)** **(i)** $t = -2$

(ii) $(3, 5, 2)$

(b) **(i)** $\overrightarrow{CA} = 12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$

(ii) $\overrightarrow{CB} = -3\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$

(iii) 42.7

(c) $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + u\begin{pmatrix} -44 \\ 60 \\ 42 \end{pmatrix}$

(d) $d = -\dfrac{713}{11}$

(e) **(i)** $70\mathbf{i} - 100\mathbf{j} - \dfrac{735}{11}\mathbf{k}$

(ii) 1980

35.2 **(a)** Refer to solution

(b) $\theta = 30^\circ$

(c) **(i)** $\overrightarrow{CA} = 2\mathbf{i}$

(ii) $\overrightarrow{CB} = \mathbf{i} + \sqrt{3}\mathbf{j}$

(iii) $A\hat{C}B = 60^\circ$

(d) 1.73

(e) **(i)** $h = 10$

(ii) 17.3

35.3 **(a)** **(i)** Refer to solution

(ii) $(4, 6, 4)$

(b) $a = 3$

(c) **(i)** $\overrightarrow{OR} = 3\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

(ii) $(-1, 0, 2)$

(d) **(i)** $\overrightarrow{PQ} = -5\mathbf{i} - 6\mathbf{j} - 2\mathbf{k}$

(ii) $-30\mathbf{i} - 36\mathbf{j} - 12\mathbf{k}$

(iii) $\overrightarrow{PR} = -\mathbf{i} - 2\mathbf{j} - \mathbf{k}$

(iv) $\overrightarrow{PS} = -14\mathbf{i} - 28\mathbf{j} - 14\mathbf{k}$

(v) 223

35.4 **(a)** **(i)** $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix}$

(ii) $-3t\mathbf{i} - 3t\mathbf{j} + (3 - 3t)\mathbf{k}$

(iii) Refer to solution

(b) **(i)** $\overrightarrow{BA} = -3\mathbf{k}$, $\overrightarrow{BC} = -3\mathbf{i}$

(ii) $\mathbf{n}_1 = -9\mathbf{i} + 9\mathbf{j}$

(iii) $\mathbf{n}_2 = -9\mathbf{j} + 9\mathbf{k}$

(iv) 60°

(c) $(0, 5, 0)$ and $(0, -5, 0)$

Chapter 9

Exercise 36

36.1 **(a)** 3

(b) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(c) 6

36.2 **(a)** 5

(b) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$

(c) 236

36.3 **(a)** **(i)** 2

(ii) 1

(b) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

(c) 4

36.4 **(a)** **(i)** 1
(ii) 2

(b) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

(c) 2

Exercise 37

37.1 **(a)** BC, DE
(b) BC, DE, AB and BD
(c) 58

37.2 **(a)** CD, EG
(b) 44
(c) CD, EG, AB, AF, CE and AC
(d) 210

37.3 **(a)** $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$

(b) AB
(c) AB, EF, DE, CD and BE
(d) 105

37.4 **(a)** $\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

(b) AE
(c) AE, CD, FG, AB, DE, EF and EH
(d) 335

Exercise 38

38.1 **(a)** AB
(b) AE, DE, BD and CD
(c) 88

38.2 **(a)** 37
(b) BE, CE, AB, AF, FG and CD
(c) 193

38.3 **(a)** $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

(b) EF, CE, CD, AF and AB
(c) 139

38.4 **(a)** $\mathbf{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

(b) DE
(c) HI, FI, DF, BD, FG, AH, EF and BC
(d) 600

Exercise 39

39.1 **(a)** Refer to solution
(b) **(i)** 5
(ii) 2

(c) $\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

(d) 33
(e) EF, FA, AB, BC, CD, DE, EA, AC, CE, EB and BE
(f) 355

39.2 (a) (i) 6

(ii) 2

(iii) 5

(b) AE

(c) AE, AF, GF, AC, BC and AD

(d) 296

(e) FG, GA, AB, BC, CD, DA, AE, EF, FA, AC, CA and AF

(f) 638

39.3 (a) (i) 3

(ii) 4

(iii) 3

(iv) 15 minutes

(b) CD, DE, EF, FA, AB, BC, CA, AG, GD, DE, EG, GB and BC

(c) 183 minutes

(d) (i) B

(ii) CD, DE, EF, FA, AB, BC, CA, AG, GD, DE, EG and GB

(iii) 178 minutes

39.4 (a) (i) 5

(ii) 2

(iii) 7

(iv) 170 seconds

(b) DE, EF, FG, GH, HA, AB, BC, CD, DB, BI, ID, DF, FI, IH and HF

(c) 1330 seconds

(d) (i) DE, EF, FG, GH, HA, AB, BC, CD, DB, BI, ID, DF, FI, IH, HF and FD

(ii) 1445 seconds

Exercise 40

40.1 (a) (i) 4

(ii) A, D

(b) BD

(c) Eulerian trail exists as there are only two vertices of odd degrees.

(d) (i) 70

(ii) 90

(e) 290

(f) 226

40.2 (a) Eulerian trail does not exist as there are more than two vertices of odd degrees.

(b) $$\mathbf{M} = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(c) 12

(d) CD, AB, AE, AC and EF

(e) 205

(f) 290

(g) 255

40.3 (a) (i) 4

(ii) 2

(b) CD, DE, EF, FA, AF, FA, AB, BC, CD and DC

(c) 159

(d) 78

(e) 78

40.4 (a) Eulerian circuit does not exist as not all vertices are of even degree.

(b) AG

(c) AG, AE, AB, AD, CD and EF

(d) 490

(e) 792

(f) 621

Chapter 10

Exercise 41

41.1 **(a)** $f'(x) = 3(\cos x - x\sin x)$
(b) **(i)** 14.1
(ii) –0.0707

41.2 **(a)** $f'(x) = -3e^{-3x}$
(b) **(i)** –2.22
(ii) 0.450

41.3 **(a)** $f'(x) = -2x\sin(x^2)$
(b) $a = 1.25$

41.4 **(a)** $g'(x) = 2x\ln x + x$
(b) $a = 1$

Exercise 42

42.1 **(a)** $f'(x) = 3\cos 3x$
(b) **(i)** –3
(ii) $y = -3x + 3\pi$

42.2 **(a)** $f'(x) = \frac{3}{x}$
(b) **(i)** 3
(ii) $-\frac{1}{3}$
(iii) $y = -\frac{1}{3}x + \frac{1}{3}$

42.3 **(a)** $f'(x) = 3e^{3x}$
(b) $3e^{3k}$
(c) $k = 1$

42.4 **(a)** $f'(x) = \frac{1}{2x}$
(b) $\frac{1}{4}$
(c) $\ln\sqrt{2} + 8$

Exercise 43

43.1 **(a)** $f'(x) = 9 + 6x - 3x^2$
(b) $x = -1$ or $x = 3$
(c) **(i)** $f''(x) = 6 - 6x$
(ii) –1
(iii) –4

43.2 **(a)** $f'(x) = -xe^{-\frac{1}{2}x^2}$
(b) $x = 0$
(c) **(i)** Refer to solution
(ii) 1

43.3 **(a)** $f'(x) = \frac{4 - 4x}{(x+1)^3}$
(b) $x = 1$
(c) **(i)** Refer to solution
(ii) 1

43.4 **(a)** $f'(x) = 2\cos\left(2x - \frac{\pi}{3}\right)$
(b) $x = -0.262$ or $x = 1.31$
(c) **(i)** $-4\sin\left(2x - \frac{\pi}{3}\right)$
(ii) –0.262
(iii) –4

Exercise 44

44.1 **(a)** $n(0) = 300$
(b) **(i)** $\frac{dn}{dt} = 84e^{0.28t}$
(ii) 451 per month
(c) 9

44.2 **(a)** $V(8) = 6 \text{ cm}^3$
(b) **(i)** $\frac{dV}{dt} = -\frac{t}{\sqrt{100 - t^2}}$
(ii) $0.75 \text{ cm}^3\text{s}^{-1}$
(c) 6

44.3 **(a)** 993
(b) $\frac{dp}{dt} = 500\cos(2t + 3.9)$
(c) 13

44.4 (a) 1059

(b) $\frac{dw}{dt} = -72.5\sin(0.5t - 5.2)$

(c) 218

Exercise 45

45.1 (a) 5.01 cm

(b) (i) $2t\cos t - t^2 \sin t$

(ii) 1.08 s

(iii) -3.39 cms^{-2}

45.2 (a) 5.65 cm

(b) (i) 1.57 s

(ii) $4\cos t - 4t\sin t$

(iii) -8.00 cms^{-2}

45.3 (a) $v(t) = 1 + e^t \cos(e^t)$

(b) (i) 2.39 s

(ii) 117 cms^{-2}

45.4 (a) 6.28 s

(b) (i) 8.64 s

(ii) $e^t(\sin t + \cos t)$

(iii) -7990 cms^{-2}

Exercise 46

46.1 (a) (i) $\pi - \theta$

(ii) $P = 16\sin\theta$

(b) (i) $\frac{dP}{d\theta} = 16\cos\theta$

(ii) $\theta = 1.57$ rad

(iii) $\theta = 1.57$ rad

(iv) 16

(c) $\theta = 0,\ \theta = \pi$

46.2 (a) (i) $CD = 10\sin\theta$, $OD = 10\cos\theta$

(ii) $400\sin\theta\cos\theta$

(iii) $100\pi - 400\sin\theta\cos\theta$

(b) (i) $400(\sin^2\theta - \cos^2\theta)$

(ii) $\theta = 0.785$ rad

(iii) $\theta = 0.785$ rad

(iv) 114

(c) $\theta = 0,\ \theta = \frac{\pi}{2}$

46.3 (a) $t = 0$ or $t = 6$

(b) (i) $Q'(t) = 3t^2 - 24t + 36$

(ii) $t = 2$ or $t = 6$

(iii) $t = 2$

(iv) 32

(c) (i) -20

(ii) $t = 0,\ t = 6$

46.4 (a) $t = 12$

(b) (i) $P'(t) = -3t^2 + 18t - 24$

(ii) $t = 2$ or $t = 4$

(iii) $t = 2$

(iv) 700

(c) (i) 704

(ii) 720

(d) 3

Chapter 11

Exercise 47

47.1 (a) $f(x) = \int \cos^3 2x \sin 2x dx$

(b) $f(x) = -\frac{1}{8}\cos^4 2x + C$

(c) $f(x) = -\frac{1}{8}\cos^4 2x + \frac{25}{8}$

47.2 (a) $f(x) = \int 2x\sin(x^2)dx$

(b) $f(x) = -\cos(x^2) + C$

(c) $f(x) = -\cos(x^2)$

47.3 (a) $f(x)=\int 3x^2(x^3+1)^6\,dx$
(b) $f(x)=\frac{1}{7}(x^3+1)^7+C$
(c) $f(x)=\frac{1}{7}(x^3+1)^7+2$

47.4 (a) $f(x)=e^{x^4}+C$
(b) $f(x)=e^{x^4}-1$

Exercise 48

48.1 (a) Refer to solution
(b) 4.88 m

48.2 (a) Refer to solution
(b) $2.27\ \text{ms}^{-1}$

48.3 (a) $v(t)=\ln(t^2+1)+C$
(b) $v(t)=\ln(t^2+1)$
(c) Refer to solution

48.4 (a) $s(t)=\sin(t^2)+C$
(b) $s(t)=\sin(t^2)+1$
(c) Refer to solution

Exercise 49

49.1 (a) 0.5 and 1.5
(b) 0.637

49.2 (a) 1
(b) 0.368

49.3 (a) −0.637 and 0.637
(b) 0.814

49.4 (a) 4, 7 and 10
(b) 40.5

Exercise 50

50.1 (a) 0.648
(b) 2.66

50.2 (a) 9.55
(b) 544

50.3 (a) $x=-3,\ x=3$
(b) $y^2=4-\frac{4x^2}{9}$
(c) 50.3

50.4 (a) $x=3-\frac{1}{2}y$
(b) 8
(c) 54.5

Exercise 51

51.1 (a) (i) 0
(ii) e^6+2
(b) 3
(c) 81200

51.2 (a) (i) 1
(ii) $2e^2$
(b) $x=7.39$
(c) 284

51.3 (a) (i) 1
(ii) $y=-0.950$
(b) 0
(c) 3.55

51.4 (a) (i) 1
(ii) 19
(b) $x=2y-2$
(c) 3
(d) 168

Exercise 52

52.1 (a) (i) $f'(a)=\frac{1}{4}$
(ii) $a=4$
(iii) $b=\frac{9}{2}$
(iv) $y=-4x+18$
(b) $\frac{35}{6}$

52.2 **(a)** **(i)** $\frac{a^2}{a-1}$

(ii) $a=2$

(iii) $y=4x-4$

(b) $\frac{2}{3}$

52.3 **(a)** $f'(h)=\frac{1}{2\sqrt{h}}$

(b) **(i)** $h=9$

(ii) $y=\frac{1}{6}x+\frac{3}{2}$

(iii) $b=-9$

(c) 9

52.4 **(a)** $f'(h)=3h^2$

(b) **(i)** $h=-\frac{1}{2}$

(ii) $-\frac{4}{3}$

(iii) $y=-\frac{4}{3}x-\frac{19}{24}$

(iv) $b=-\frac{19}{32}$

(c) 0.0215

Exercise 53

53.1 **(a)** 64 ms^{-1}

(b) $t=7$

(c) 84.2 m

(d) $a(t)=-3(t-4)^2$

(e) $4<t\le 8$

(f) $s(t)=-\frac{1}{4}(t-4)^4+92$

53.2 **(a)** $-\frac{1}{2}t^4+4t^3-12t^2+16t$

(b) 6.57195 m

(c) 9.43 m

(d) $a(t)=-6t^2+24t-24$

(e) $0\le t<2$

53.3 **(a)** **(i)** Refer to solution

(ii) $s(t)=\sin \pi t+1$

(b) $t=\frac{7}{2}$

(c) **(i)** $a(t)=-\pi^2 \sin \pi t$

(ii) $1<t<2$ or $3<t<4$

(d) 5

53.4 **(a)** $t^4-11t^3+44t^2-76t+48$

(b) $\frac{1}{5}t^5-\frac{11}{4}t^4+\frac{44}{3}t^3$
$-38t^2+48t$

(c) $0<t<2$ or $2.61<t<3.64$

(d) **(i)** 3

(ii) 2

Chapter 12

Exercise 54

54.1 **(a)** $\int \frac{1}{x}\,dx=\int \pi^2 \sin \pi t\,dt$

(b) $\ln x=-\pi\cos \pi t+C$

(c) $x=e^{-\pi\cos \pi t}$

54.2 **(a)** $\int \frac{1}{y}\,dy=\int \frac{1}{2x+3}\,dx$

(b) $\ln y=\frac{1}{2}\ln(2x+3)+C$

(c) $y=e^{\frac{1}{2}\ln(2x+3)+\ln 4}$

54.3 **(a)** $\frac{1}{3}y^3=2x+C$

(b) $y=\sqrt[3]{6x+3}$

(c) $y=5$

54.4 **(a)** $\frac{1}{2y^2}=e^x+C$

(b) $y=\frac{1}{\sqrt{2e^x+12}}$

(c) $y=\frac{1}{6}$

Exercise 55

55.1 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I})=\lambda^2-8\lambda-9$

(b) $\lambda_1=-1,\ \lambda_2=9$

(c) $\mathbf{v}_1=\begin{pmatrix}-1\\1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}\frac{3}{2}\\1\end{pmatrix}$

(d) $\mathbf{X}=Ae^{-t}\begin{pmatrix}-1\\1\end{pmatrix}+Be^{9t}\begin{pmatrix}\frac{3}{2}\\1\end{pmatrix}$

55.2 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I})=\lambda^2-3\lambda-28$

(b) $\lambda_1=-4,\ \lambda_2=7$

(c) $\mathbf{v}_1=\begin{pmatrix}-\frac{5}{2}\\1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}3\\1\end{pmatrix}$

(d) **(i)** $x=-\frac{5}{2}Ae^{-4t}+3Be^{7t}$

(ii) $y=Ae^{-4t}+Be^{7t}$

55.3 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I})=\lambda^2-2\lambda+10$

(b) $1+3\mathrm{i},\ 1-3\mathrm{i}$

(c) $\mathbf{X}=e^{t}(Ae^{3t\mathrm{i}}\mathbf{v}_1+Be^{-3t\mathrm{i}}\mathbf{v}_2)$

(d) The phase portrait is an unstable spiral.

55.4 **(a)** $\det(\mathbf{M}-\lambda\mathbf{I})=\lambda^2+10\lambda+29$

(b) $-5+2\mathrm{i},\ -5-2\mathrm{i}$

(c) $\mathbf{X}=e^{-5t}(Ae^{2t\mathrm{i}}\mathbf{v}_1+Be^{-2t\mathrm{i}}\mathbf{v}_2)$

(d) The phase portrait is a stable spiral.

Exercise 56

56.1 **(a)** $y_1=10.6$

(b) **(i)** $y_2=11.12639$

(ii) $y_3=11.7$

56.2 **(a)** $y_1=3.8$

(b) **(i)** $y_2=7.084281$

(ii) $y_3=13.050070$

(iii) $y_4=23.9$

56.3 **(a)** $y_1=2.75$

(b) **(i)** $y_2=3.428$

(ii) $y_3=4.0734$

(iii) $y_4=4.70$

56.4 **(a)** $y_1=4.2$

(b) **(i)** $y_2=6.248$

(ii) $y_3=9.8166$

(iii) $y_4=16.1646$

(iv) $y_5=27.7$

Exercise 57

57.1 **(a)** **(i)** $x_1=6.05$

(ii) $y_1=3.05$

(b) **(i)** $x_2=7.48$

(ii) $y_2=4.55$

57.2 **(a)** **(i)** $x_1=1.11$

(ii) $y_1=1.01$

(b) **(i)** $x_2=1.12$

(ii) $y_2=1.02$

(iii) $x_3=1.13$

(iv) $y_3=1.03$

57.3 **(a)** $\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=-8u-15x\\ \frac{\mathrm{d}x}{\mathrm{d}t}=u\end{cases}$

(b) **(i)** $u_1=0.2$

(ii) $x_1=0.1$

(c) $x_2=0.12$

57.4 **(a)** $\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t}=12u-36x\\ \frac{\mathrm{d}x}{\mathrm{d}t}=u\end{cases}$

(b) **(i)** $u_1=1.2$

(ii) $x_1=2.15$

(c) $x_2=2.21$

Exercise 58

58.1 **(a)** $(0, 0)$, $(6, 1.5)$

(b) **(i)** $\frac{dy}{dx} = \frac{(x-6)y}{(3-2y)x}$

(ii) $\left.\frac{dy}{dx}\right|_{t=0} = 2.5$

(iii) The population of shark is decreasing at the beginning.

58.2 **(a)** $(0, 0)$, $(8, 0.4)$

(b) **(i)** $\frac{dy}{dx} = \frac{(x-8)y}{(2-5y)x}$

(ii) $\left.\frac{dy}{dx}\right|_{t=0} = \frac{3}{13}$

(iii) $0 \le y < 0.4$

58.3 **(a)** The population of crocodile is increasing at the beginning.

(b) **(i)** 600

(ii) 550

58.4 **(a)** $x > \frac{10}{3}$

(b) **(i)** 210

(ii) 270

Exercise 59

59.1 **(a)** $(0, 0)$

(b) $\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^2 + 8\lambda + 12$

(c) $\lambda_1 = -6$, $\lambda_2 = -2$

(d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

(e) **(i)** $x = -e^{-6t} + 10e^{-2t}$

(ii) $y = -e^{-6t} + 2e^{-2t}$

(f) (1.35, 0.268)

(g) Refer to solution

59.2 **(a)** $y > 3$

(b) $\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^2 - \lambda - 12$

(c) $\lambda_1 = -3$, $\lambda_2 = 4$

(d) $\mathbf{v}_1 = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(e) **(i)** $x = 7e^{-3t}$

(ii) $y = e^{-3t} + e^{4t}$

(f) The population of horse will approach 0.

(g) Refer to solution

59.3 **(a)** $\begin{cases} \frac{dv}{dt} = 3v + 4x \\ \frac{dx}{dt} = v \end{cases}$

(b) **(i)** $v_1 = 1.3$

(ii) $x_1 = 0.1$

(c) **(i)** 0.23 m

(ii) 0.403 m

(d) $\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^2 - 3\lambda - 4$

(e) $\lambda_1 = -1$, $\lambda_2 = 4$

(f) $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$

(g) **(i)** $x = -0.2e^{-t} + 0.2e^{4t}$

(ii) 0.516 m

(h) 0.113 m

59.4 **(a)** $\begin{cases} \dfrac{dv}{dt} = 9x \\ \dfrac{dx}{dt} = v \end{cases}$

(b) **(i)** $v_1 = 2.25$

(ii) $x_1 = 1$

(c) **(i)** 1.56 m

(ii) 2.69 m

(iii) 4.69 m

(d) $\det(\mathbf{M} - \lambda\mathbf{I}) = \lambda^2 - 9$

(e) $\lambda_1 = -3$, $\lambda_2 = 3$

(f) $\mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(g) **(i)** $x = 0.5e^{-3t} + 0.5e^{3t}$

(ii) 10.1 m

(h) 53.4%

Chapter 13

Exercise 60

60.1 **(a)** $\lambda^2 - 0.75\lambda - 0.25$

(b) $\lambda_1 = -\dfrac{1}{4}$, $\lambda_2 = 1$

(c) $\mathbf{v} = \begin{pmatrix} \dfrac{6}{25} \\ \dfrac{19}{25} \end{pmatrix}$

60.2 **(a)** $\lambda^2 - 1.35\lambda + 0.35$

(b) $\lambda_1 = \dfrac{7}{20}$, $\lambda_2 = 1$

(c) $\mathbf{v} = \begin{pmatrix} \dfrac{3}{13} \\ \dfrac{10}{13} \end{pmatrix}$

60.3 **(a)** $\lambda_1 = -\dfrac{73}{100}$, $\lambda_2 = 1$

(b) $\mathbf{v}_8 = \begin{pmatrix} 0.519 \\ 0.481 \end{pmatrix}$

(c) $\mathbf{v} = \begin{pmatrix} \dfrac{90}{173} \\ \dfrac{83}{173} \end{pmatrix}$

60.4 **(a)** $\lambda_1 = -\dfrac{11}{100}$, $\lambda_2 = 1$

(b) $\mathbf{v}_{13} = \begin{pmatrix} 0.568 \\ 0.432 \end{pmatrix}$

(c) $\mathbf{v} = \begin{pmatrix} \dfrac{21}{37} \\ \dfrac{16}{37} \end{pmatrix}$

Exercise 61

61.1 **(a)** $\mathbf{T} = \begin{pmatrix} 0.7 & 0.25 \\ 0.3 & 0.75 \end{pmatrix}$

(b) $\lambda^2 - 1.45\lambda + 0.45$

(c) $\lambda_1 = \dfrac{9}{20}$, $\lambda_2 = 1$

(d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ \dfrac{6}{5} \end{pmatrix}$

(e) **(i)** $\begin{pmatrix} 1 & 1 \\ -1 & \dfrac{6}{5} \end{pmatrix}$

(ii) $\begin{pmatrix} \left(\dfrac{9}{20}\right)^n & 0 \\ 0 & 1 \end{pmatrix}$

(f)

$$\begin{pmatrix} \dfrac{6}{11}\left(\dfrac{9}{20}\right)^n + \dfrac{5}{11} & -\dfrac{5}{11}\left(\dfrac{9}{20}\right)^n + \dfrac{5}{11} \\ -\dfrac{6}{11}\left(\dfrac{9}{20}\right)^n + \dfrac{6}{11} & \dfrac{5}{11}\left(\dfrac{9}{20}\right)^n + \dfrac{6}{11} \end{pmatrix}$$

(g) 3810

61.2

(a) $\mathbf{T}=\begin{pmatrix}0.6 & 0.36\\ 0.4 & 0.64\end{pmatrix}$

(b) $\lambda^2-1.24\lambda+0.24$

(c) $\lambda_1=\frac{6}{25},\ \lambda_2=1$

(d) $\mathbf{v}_1=\begin{pmatrix}1\\ -1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\ \frac{10}{9}\end{pmatrix}$

(e) **(i)** $\begin{pmatrix}1 & 1\\ -1 & \frac{10}{9}\end{pmatrix}$

(ii) $\begin{pmatrix}\left(\frac{6}{25}\right)^n & 0\\ 0 & 1\end{pmatrix}$

(f)

$$\begin{pmatrix}\frac{10}{19}\left(\frac{6}{25}\right)^n+\frac{9}{19} & -\frac{9}{19}\left(\frac{6}{25}\right)^n+\frac{9}{19}\\ -\frac{10}{19}\left(\frac{6}{25}\right)^n+\frac{10}{19} & \frac{9}{19}\left(\frac{6}{25}\right)^n+\frac{10}{19}\end{pmatrix}$$

(g) 0.526

61.3

(a) $\mathbf{T}=\begin{pmatrix}0.82 & 0.13\\ 0.18 & 0.87\end{pmatrix}$

(b) $\lambda_1=\frac{69}{100},\ \lambda_2=1$

(c) $\mathbf{v}_1=\begin{pmatrix}1\\ -1\end{pmatrix},\ \mathbf{v}_2=\begin{pmatrix}1\\ \frac{18}{13}\end{pmatrix}$

(d) **(i)** $\begin{pmatrix}1 & 1\\ -1 & \frac{18}{13}\end{pmatrix}$

(ii) $\begin{pmatrix}\left(\frac{69}{100}\right)^n & 0\\ 0 & 1\end{pmatrix}$

(e)

$$\begin{pmatrix}\frac{18}{31}\left(\frac{69}{100}\right)^n+\frac{13}{31} & -\frac{13}{31}\left(\frac{69}{100}\right)^n+\frac{13}{31}\\ -\frac{18}{31}\left(\frac{69}{100}\right)^n+\frac{18}{31} & \frac{13}{31}\left(\frac{69}{100}\right)^n+\frac{18}{31}\end{pmatrix}$$

(f) $\begin{pmatrix}50\left(\frac{69}{100}\right)^n+260\\ -50\left(\frac{69}{100}\right)^n+360\end{pmatrix}$

(g) **(i)** 260

(ii) 360

61.4 **(a)** $\mathbf{T} = \begin{pmatrix} 0.9 & 0.07 \\ 0.1 & 0.93 \end{pmatrix}$

(b) $\lambda_1 = \frac{83}{100}$, $\lambda_2 = 1$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ \frac{10}{7} \end{pmatrix}$

(d) **(i)** $\begin{pmatrix} 1 & 1 \\ -1 & \frac{10}{7} \end{pmatrix}$

(ii) $\begin{pmatrix} \left(\frac{83}{100}\right)^n & 0 \\ 0 & 1 \end{pmatrix}$

(e)

$$\begin{pmatrix} \frac{10}{17}\left(\frac{83}{100}\right)^n + \frac{7}{17} & -\frac{7}{17}\left(\frac{83}{100}\right)^n + \frac{7}{17} \\ -\frac{10}{17}\left(\frac{83}{100}\right)^n + \frac{10}{17} & \frac{7}{17}\left(\frac{83}{100}\right)^n + \frac{10}{17} \end{pmatrix}$$

(f) $\begin{pmatrix} \frac{16}{85}\left(\frac{83}{100}\right)^n + \frac{7}{17} \\ -\frac{16}{85}\left(\frac{83}{100}\right)^n + \frac{10}{17} \end{pmatrix}$

(g) 0.988

(h) **(i)** $\frac{7}{17}$

(ii) $\frac{10}{17}$

Chapter 14

Exercise 62

62.1 **(a)** $P(X < 2) = 0.493$

(b) $P(Y = 10) = 0.125$

(c) $P(Y < 6 \mid Y < 9) = 0.193$

62.2 **(a)** $P(X > 10) = 0.731$

(b) $P(Y \le 33) = 0.217$

(c) $P(Y = 43 \mid Y \ge 40) = 0.111$

62.3 **(a)** $P(X \ge 2) = 0.815$

(b) $P(Y \le 28) = 0.923$

(c) $P(X \le 4)^7 = 0.206$

62.4 **(a)** $P(X > 7) = 0.401$

(b) $P(X_0 \le 2) = 0.744$

(c) $P(X = 5)P(Y = 5) = 0.0205$

Exercise 63

63.1 **(a)** $b = 4$

(b) **(i)** $P(X = 2) = 0.147$

(ii) 4

63.2 **(a)** $c = 9$

(b) **(i)** $P(X \le 4) = 0.0550$

(ii) 3

63.3 **(a)** $\lambda = 15$

(b) $P(Y > 3) = 0.170$

63.4 **(a)** $\lambda = 0.0100$

(b) $P(Y > 2) = 0.120$

Chapter 15

Exercise 64

64.1 **(a)** $E(1 - 10X) = -214$

(b) $Var(5 + 2X) = 12$

(c) $E(5 + 4X + 3Y) = 151$

64.2 **(a)** $E(X) = 8$

(b) $Var(X) = 1$

(c) $E(-X - Y) = -28$

64.3 **(a)** $E(100 - X) = 105$

(b) $Var(99 - 5X) = 400$

(c) $Var(6X - 5Y) = 776$

64.4 **(a)** $E(X) = -3$

(b) $Var(X) = 3$

(c) $Var(10Y - 3X) = 477$

Exercise 65

65.1 **(a)** **(i)** 640 g
(ii) 256 g
(b) **(i)** 760 g
(ii) 20.4 g
(c) $P(Y < 795) = 0.957$

65.2 **(a)** **(i)** 1068 cm
(ii) 54 cm
(b) $P(1070 < X < 1090) = 0.391$
(c) $P(Y > 330) = 0.995$

65.3 **(a)** $P(X + Y > 1000) = 0.0805$
(b) $P(X - Y > 0) = 0.920$

65.4 **(a)** 0.367
(b) 0.0368

Exercise 66

66.1 **(a)** $E(X + Y) = 9$
(b) $P(5 < X + Y < 8) = 0.208$
(c) $Var(X - 4Y) = 61.5$

66.2 **(a)** $E(W + X + Y) = 34$
(b) $P(W + X + Y \geq 30) = 0.776$
(c) $Var(3W - 2X - Y) = 184$

66.3 **(a)** 39.5
(b) 0.0627
(c) 6.10×10^{-8}

66.4 **(a)** 375
(b) 0.0205
(c) 8.67×10^{-6}

Exercise 67

67.1 **(a)** $E(X) = 3.7$
(b) $E(5X + 2Y) = 38.5$
(c) $Var(5X + 2Y) = 140.75$

67.2 **(a)** $E(X) = 65$
(b) $Var(X) = 22.75$
(c) $E(2X - 7Y) = -80$
(d) $Var(2X - 7Y) = 434$

67.3 **(a)** $E(X) = 0.45$
(b) $Var(X) = 0.4365$
(c) $E(-X - Y) = 0.75$
(d) $Var(-X - Y) = 1.2365$

67.4 **(a)** **(i)** $\frac{1}{5}$
(ii) $P(X = 0) = \frac{3}{5}$
(iii) $E(X) = 6$
(b) $E(Y) = \frac{1}{3}$

Chapter 16

Exercise 68

68.1 **(a)** 300
(b) $Var(\bar{X}) = \frac{1}{40}$
(c) $P(\bar{X} < 299.85) = 0.171$

68.2 **(a)** -2
(b) $\frac{1}{2}$
(c) $P(|\bar{X}| < 1.5) = 0.159$

68.3 **(a)** 5
(b) $Var(\bar{X}) = \frac{4}{125}$
(c) $P(\bar{X} + \bar{Y} > 0.1) = 0.324$

68.4 **(a)** $n = 108$
(b) $P(9 < \bar{X} - \bar{Y} < 21) = 0.958$

Exercise 69

69.1 **(a)** $s_{n-1} = 5.09$
(b) 59

69.2 **(a)** $s_n^2 = 1.2$
(b) 195

69.3 **(a)** $s_n = 2.35$
(b) -228

69.4 **(a)** **(i)** 16.6
(ii) 3.3
(b) $n = 110$

Chapter 17

Exercise 70

70.1 **(a)** 401 g
(b) $s_{n-1} = 103$ g
(c) (308.33, 493.67)

70.2 **(a)** 54 s
(b) (43.442, 64.558)

70.3 **(a)** $n = 25$
(b) (19.013, 20.987)
(c) 1.974

70.4 **(a)** $n = 16$
(b) (68.685, 71.315)
(c) 2.63

Exercise 71

71.1 **(a)** 20.65
(b) $\sigma = 2.64$

71.2 **(a)** 3.18
(b) $s_{n-1} = 0.448$

71.3 **(a)** A confidence interval with a smaller confidence level has a narrower interval about the mean.
(b) (36.4, 38.6)
(c) $s_{n-1} = 2.06$

71.4 **(a)** A confidence interval with a smaller confidence level has a narrower interval about the mean.
(b) (191.6, 204.4)
(c) $\sigma = 7.45$

Chapter 18

Exercise 72

72.1 **(a)** **(i)** $a = 6226$, $b = 2677$
(ii) 30700 km
(b) **(i)** $r = 0.865$
(ii) $R^2 = 0.748$
(iii) 74.8% of the variability of the data is explained by the regression model.

72.2 **(a)** **(i)** $a = -0.132$, $b = 8.09$
(ii) 4.79 g
(b) **(i)** $r = -0.773$
(ii) $R^2 = 0.597$
(iii) 59.7% of the variability of the data is explained by the regression model.

72.3 **(a)** **(i)** $a = 0.533$, $b = 13.9$
(ii) a represents the increase in expenditure when the time spent in a market increases by 1 minute.
(b) **(i)** $r = 0.479$
(ii) $R^2 = 0.229$
(iii) 22.9% of the variability of the data is explained by the regression model.

72.4 **(a)** **(i)** $a = 0.11$, $b = -0.26$
(ii) b represents the annual corn yield rate when the annual rainfall is 0 cm.
(b) **(i)** $r = 0.957$
(ii) $R^2 = 0.917$
(iii) 91.7% of the variability of the data is explained by the regression model.

Exercise 73

73.1 **(a)** **(i)** 1050 USD
(ii) 810 USD
(iii) 450 USD
(b) $SS_{res} = 33900$
(c) 87.2% of the variability of the data is explained by the regression model.

73.2 **(a)** **(i)** 74.6625 marks
(ii) 78.9625 marks
(iii) 74.0625 marks
(b) $SS_{res} = 131$
(c) 63.2% of the variability of the data is explained by the regression model.

73.3 **(a)** **(i)** 9231
(ii) 7200
(iii) 11264
(b) $SS_{res} = 60000$
(c) Model 1

73.4 **(a)** **(i)** 212
(ii) 192
(iii) 155
(b) $SS_{res} = 2080$
(c) Model 2

Exercise 74

74.1 **(a)** **(i)** $y = 0.00867x^3 - 3.92x^2 + 590x - 29500$
(ii) 59.9 kg
(b) **(i)** $R^2 = 0.967$
(ii) 96.7% of the variability of the data is explained by the regression model.

74.2 **(a)** **(i)** $y = 23.8x^{0.169}$
(ii) 24.7 g
(b) **(i)** $R^2 = 0.360$
(ii) 36.0% of the variability of the data is explained by the regression model.

74.3 **(a)** **(i)** $y = 0.25x^2 + 2.25x - 1$
(ii) $R^2 = 0.967$
(b) $SS_{res} = 1$
(c) $SS_{tot} = 30.0$

74.4 **(a)** **(i)** $y = 2.37 \cdot 1.36^x$
(ii) $R^2 = 0.99957$
(b) $SS_{res} = 0.0136$
(c) $SS_{tot} = 31.6$

Chapter 19

Exercise 75

75.1 **(a)** H_0: The data follows a Poisson distribution with mean 2.
(b) 17.1
(c) 4
(d) 55.1
(e) The null hypothesis is rejected as $\chi^2_{calc} > 7.779$.

75.2 **(a)** H_0: The data follows a Binomial distribution with parameters $B(5, 0.6)$.
(b) **(i)** 1.024
(ii) 4
(c) 24.8
(d) The null hypothesis is rejected as $\chi^2_{calc} > 9.488$.

75.3 **(a)** H_0: The data follows a Normal distribution with parameters $N(13, 4)$.
(b) **(i)** 4.01
(ii) 2
(c) $p\text{-value} = 0.407$
(d) The null hypothesis is not rejected as $p\text{-value} > 0.05$.

75.4 **(a)** H_0: The data follows a Normal distribution with parameters $N(3.4, 0.15)$.
(b) 18.2
(c) 3
(d) $p\text{-value} = 0.0000376$
(e) The null hypothesis is rejected as $p\text{-value} < 0.01$.

Exercise 76

76.1 **(a)** **(i)** $H_0: \mu = 17$
(ii) $H_1: \mu \neq 17$
(b) $p\text{-value} = 0.0456$
(c) The null hypothesis is rejected as $p\text{-value} < 0.05$.

76.2 **(a)** **(i)** $H_0: \mu = 28$
(ii) $H_1: \mu < 28$
(b) $p\text{-value} = 0.412$
(c) The null hypothesis is not rejected as $p\text{-value} > 0.1$.

76.3 **(a)** **(i)** $H_0: \mu = 7$
(ii) $H_1: \mu > 7$
(b) **(i)** 7.1875
(ii) $z = 0.433$
(c) The null hypothesis is not rejected as $z < 1.645$.

76.4 **(a)** **(i)** $H_0: \mu = 100$
(ii) $H_1: \mu \neq 100$
(b) **(i)** 106
(ii) $z = 5.55$
(c) The null hypothesis is rejected as $|z| > 2.326$.

Exercise 77

77.1 **(a)** $H_1: \mu_d < 0$
(b) $p\text{-value} = 0.0122$
(c) -2.99
(d) The null hypothesis is not rejected as $p\text{-value} > 0.01$.

77.2 **(a)** $H_1: \mu_d > 0$
(b) $p\text{-value} = 0.133$
(c) 1.21
(d) The null hypothesis is not rejected as $p\text{-value} > 0.05$.

77.3 **(a)** **(i)** $H_0: \mu_d = 0$
(ii) $H_1: \mu_d < 0$
(b) $p\text{-value} = 0.222$
(c) The null hypothesis is not rejected as $p\text{-value} > 0.1$.
(d) $\frac{4}{9}$

77.4 **(a)** **(i)** $H_0: \mu_d = 0$
(ii) $H_1: \mu_d > 0$
(b) $p\text{-value} = 0.0198$
(c) The null hypothesis is rejected as $p\text{-value} < 0.05$.
(d) $\frac{2}{3}$

Exercise 78

78.1 **(a)** **(i)** $H_0: p = 0.15$
(ii) $H_1: p < 0.15$
(b) 0.00426
(c) The null hypothesis is rejected as $p\text{-value} < 0.01$.

78.2 **(a)** **(i)** $H_0: \lambda = 10$
(ii) $H_1: \lambda > 10$
(b) 0.208
(c) The null hypothesis is not rejected as $p\text{-value} > 0.05$.

78.3 **(a)** **(i)** $H_0:\ p = 0.03$

(ii) $H_1:\ p > 0.03$

(b) **(i)** 7

(ii) 6

78.4 **(a)** **(i)** $H_0:\ \lambda = 2.5$

(ii) $H_1:\ \lambda > 2.5$

(b) **(i)** 8

(ii) 7

Exercise 79

79.1 **(a)** **(i)** $H_0:\ \lambda = 9$

(ii) $H_1:\ \lambda > 9$

(b) 0.0739

(c) 0.573

(d) $0 \le Y \le 4$

(e) 0.0367

(f) The null hypothesis is not rejected as p-value > 0.01.

79.2 **(a)** **(i)** $H_0:\ q = 0.92$

(ii) $H_1:\ q > 0.92$

(b) 0.0827

(c) 0.810

(d) $Y \ge 7$

(e) 0.185

(f) The null hypothesis is not rejected as p-value > 0.05.

79.3 **(a)** **(i)** $H_0:\ \mu = 250$

(ii) $H_1:\ \mu > 250$

(b) 0.0127

(c) 0.868

(d) $\overline{Y} \le 298$ or $\overline{Y} \ge 302$

(e) The null hypothesis is rejected as the sample mean is less than 298.

(f) $\overline{Y} \le 299$ or $\overline{Y} \ge 301$

79.4 **(a)** **(i)** $H_0:\ \mu = 100$

(ii) $H_1:\ \mu \ne 100$

(b) 0.134

(c) 0.499

(d) $\overline{Y} \le 24.1$

(e) The null hypothesis is rejected as the sample mean is less than 24.1.

(f) $\overline{Y} \le 23.7$

Exercise 80

80.1 **(a)** **(i)** $H_0:\ \rho = 0$

(ii) $H_1:\ \rho < 0$

(b) p-value $= 0.00119$

(c) The null hypothesis is rejected as p-value < 0.1.

(d) **(i)** $a = -4.24$, $b = 99.2$

(ii) b represents the expected quiz score of a student who didn't revise.

(e) 69.5 marks

(f) **(i)** $r = -0.960$

(ii) $R^2 = 0.921$

(iii) 92.1% of the variability of the data is explained by the regression model.

80.2 **(a)** **(i)** $H_0:\ \rho = 0$

(ii) $H_1:\ \rho \neq 0$

(b) $p\text{-value} = 0.0164$

(c) The null hypothesis is rejected as $p\text{-value} < 0.05$.

(d) **(i)** $a = 0.937$, $b = -5.07$

(ii) a represents the average increase in the number of webpages visited when the time for accessing the internet is increased by 1 minute.

(e) 16

(f) **(i)** $r = 0.893$

(ii) $R^2 = 0.798$

(iii) 79.8% of the variability of the data is explained by the regression model.

80.3 **(a)** **(i)** $H_0:\ \rho = 0$

(ii) $H_1:\ \rho > 0$

(b) $p\text{-value} = 0.519$

(c) The null hypothesis is not rejected as $p\text{-value} > 0.05$.

(d) This suggestion is not a valid approach as from the result of the hypothesis test, the two variables are weakly correlated.

(e) **(i)** $R^2 = 0.000861$

(ii) 0.861% of the variability of the data is explained by the regression model.

(f) $SS_{res} = 18822$

(g) $SS_{tot} = 195000$

80.4 **(a)** **(i)** $H_0:\ \rho = 0$

(ii) $H_1:\ \rho < 0$

(b) $p\text{-value} = 0.181$

(c) The null hypothesis is not rejected as $p\text{-value} > 0.1$.

(d) This suggestion is not a valid approach as from the result of the hypothesis test, the two variables are weakly correlated.

(e) **(i)** $R^2 = 0.278$

(ii) 27.8% of the variability of the data is explained by the regression model.

(f) $SS_{res} = 131.6875$

(g) $SS_{tot} = 314$

Chapter 20

Quick Practice

I **(a)** **(1)** $a = -0.748$, $b = 11.5$, $c = -57.9$, $d = 451$

(2) 354

(3) This estimation is valid as this is an interpolation.

(b) **(1)** 346.2

(2) $H_0:\ \mu = 370$

(3) $H_1:\ \mu < 370$

(4) $p\text{-value} = 0.0313$

(5) The null hypothesis is rejected as $p\text{-value} < 0.05$.

II **(c)** **(1)** $403.91 \cdot 0.97911^n$

(2) $R^2 = 0.971$

(3) 97.1% of the variability of the data is explained by the regression model.

(d) **(1)** 6

(2) 1.54

(3) The null hypothesis is not rejected as $\chi^2_{calc} < 14.449$.

III **(e)** **(1)** $H_0:\ \mu_d = 0$

(2) $H_1:\ \mu_d > 0$

(3) $p\text{-value} = 0.0584$

(4) The null hypothesis is rejected as $p\text{-value} < 0.1$.

(f) **(1)** The numbers of books sold by the two authors follow a bivariate normal distribution.

(2) $H_0:\ \rho = 0$

(3) $H_1:\ \rho \neq 0$

(4) $p\text{-value} = 0.0111$

(5) The null hypothesis is rejected as $p\text{-value} < 0.05$.

(g) **(1)** $(-0.592, 20.0)$

(2) The above result is not consistent with the conclusion of the hypothesis test in (e) as 0 is included in the confidence interval.

Exercise 81

81.1 **(a)** Refer to solution

(b) **(1)** $\mathbf{M} = \begin{pmatrix} 4 & 12 \\ 1 & 0 \end{pmatrix}$

(2) $\lambda^2 - 4\lambda - 12$

(3) $\lambda_1 = -2$, $\lambda_2 = 6$

(4) $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$

(5) $y = Ae^{-2x} + Be^{6x}$

(6) Refer to solution

(c) **(1)** The eigenvalues of $\mathbf{M}$ are the solutions of $\lambda^2 - 4\lambda - 12 = 0$, where -4 and -12 are the coefficients of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y in $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} - 12y = 0$.

(2) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + b\frac{\mathrm{d}y}{\mathrm{d}x} + cy = 0$

(d) **(1)** $y = Ae^{3x} + Be^{9x}$

(2) $y = 2e^{3x} + e^{9x}$

81.2 **(a)** **(1)** -2

(2) $\frac{1}{2}$

(3) $y = \frac{1}{2}x + \frac{9}{2}$

(4) $\left(10, \frac{19}{2}\right)$

(5) 559 m

(b) **(1)** 333 m

(2) 6450 m

(c) **(1)** 6

(2) 4

(d) Eulerian circuit does not exist as not all vertices are of even degree.

(e) **(1)** CJ

(2) CJ, HI, DE, FG, AB, BH, BC, AD and GH

(3) 265 s

(f) EF, FG, GH, HB, BC, CB, BA, AD, DA, AG, GH, HI, IJ, JC, CD and DE

(g) AD, BC, GH

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